WEAVING A UNIFORMLY THICK SHEET FROM RECTANGLES

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Abstract. In this paper, we demonstrate a way to weave together finite-length strips into a uniformly thick infinite sheet. Because we require our sheet to be locked and unable to slip, our model requires more layers than a conventional weave. For an arbitrary rectangle, the sheet is at most 16 layers thick. For some families of tileable shapes, the sheet is at most 18 layers thick. Certain specially designed shapes achieve thinner weaves. We have also designed finite weaves of rectangles that remain locked and uniformly thick, but at the cost of doubling the number of layers.

1. Introduction

Many children know how to weave a few strips of paper into a sheet, which will have uniform thickness. However, the size of the sheet in this simple weaving is limited by the length of the original strips. This sheet will also require some sort of locking mechanism to hold it together: without tape or a non-uniform edge, the strips can slide apart.

By contrast, we show how to weave together finite-length strips into an infinite sheet of uniform thickness. In addition, our sheet is locked, so the strips will not slip, assuming the creases stay folded. Formally, the folded components cannot simultaneously separate via rigid motions (i.e., treating each component as a rigid object). However, our weaving requires more layers than the child's model.

Table 1 summarizes the number of layers required by our various sheet-weaving algorithms. For a long rectangular strip (aspect ratio > 2), our infinite sheet is eight layers thick. For a nearly square rectangular strip (aspect ratio ≤ 2), our infinite sheet is sixteen layers thick. In the special case of 1×5 rectangles, we can achieve a thickness of just five layers. More broadly, we study when the "strip" is a shape other than a rectangle. For any polygon that tiles the plane, where tiles appear in only finitely many orientations, we give a method for weaving an infinite sheet eighteen layers thick. We also design a special concave polygon that can be woven into a sheet just four layers thick.

A more challenging goal is to create a locked *finite* sheet of uniform thickness, where boundary conditions come into play. If we just apply a portion of our infinite constructions, the edges of the sheet will be ragged, and locking the sheet while keeping the thickness constant proves difficult. We devise solutions to these problems that generate locked, uniform-thickness, finite woven sheets—but at the price of double the thickness, and only for certain tessellations. See Table 1, right column.

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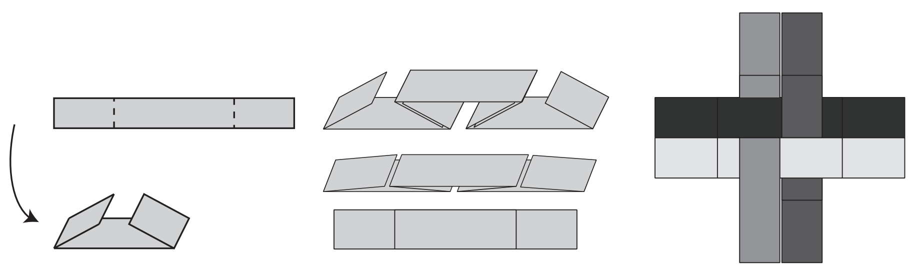
	Number of Layers		
Polygon Type	**Infinite Sheet**	**Finite Sheet**	**Section**
Rectangle, $1 \times > 2$	8	16	§2
Rectangle, $1 \times \leq 2$	16	32	§2
Parallelo-hexagon	16 or 32	32 or 64	§2
Rectangle, 1×5	5	10	§3
Tileable polygon	18*	36*	§4
Special polygon	4	8	§5

Table 1. Our results for locked sheets: paper shapes and resulting number of layers for infinite and finite weaves. *Applies only to certain tilings.

2. General Rectangles

Given a rectangle of dimensions 1 by > 2, we can create a locked infinite weave of eight layers. Refer to Figure 1. First, fold the two narrower edges of the rectangle to the middle. Next, weave rectangles together to form an infinite strip, interlacing the folded flaps with each other. The interlacing of the flaps forces the rectangles into a locked one-dimensional configuration. Because each unit is effectively two layers thick and we interlace each unit with another unit, we so far have four layers. Finally, weave these four-layer infinite strips into a standard cross-weave, giving us eight layers in total. Because each unit is at least as long as it is wide, the cross-weave will prevent the strips from separating, locking the construction in two dimensions.



(a) Forming the individual unit

(b) Lace into a strip and flatten

(c) Weave strips together

Figure 1. Constructing the general rectangle weave

This method can be extended to other shapes as well.

For rectangles of dimensions $1\times \leq 2$, we can fold the shape in half along the narrower edge. This operation doubles the thickness of the weave, but reduces to the long-rectangle case. The result is 16 layers thick. (If we tried to apply the previous construction directly to, e.g., a square, then we could still make an infinite one-dimensional strip, but the cross-weave would not prevent the strips from coming apart by sliding the units in a perpendicular direction.)

Define a *parallelo-hexagon* to be a hexagon where each side is the same length as and parallel to the opposite side, and two pairs of opposite sides are of the same length. See [de Villiers 12] for a strict definition. We can fold such a parallelo-hexagon into a rectangle, as in Figure 2. The resulting weave is twice as thick as the rectangle weave: 16 layers if the generated rectangle is of dimensions $1 \times > 2$, and 32 layers otherwise.

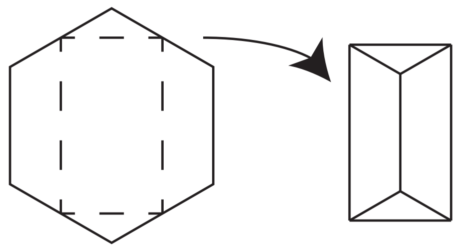


FIGURE 2. Folding a parallelo-hexagon into a rectangle.

3. $\mathbf{1 \times 5}$ Rectangle

For rectangles of specific dimensions, we have found special methods that result in thinner weaves. Take the 1×5 rectangle. As shown in Figure 3, we can achieve a locked infinite weave of only five layers.

First, instead of folding the edges of the strip to the middle, we divide the strip into fifths and fold the flaps along the one-fifth and four-fifths mark. These folds result in the extreme thirds being two layers thick and the center third being one layer thick.

As before, we interlace these units into infinite locked one-dimensional strips. These strips alternate with squares of thickness 1 and 4.

Finally, we weave the strips together, locking them both horizontally and vertically into a weave. When weaving the strips together, we align the strips so that the four-layer-thick section of one strip overlaps with the one-layer-thick section of the orthogonal strip. This matching is possible by arranging the horizontal strips to form a checkerboard pattern of 1 and 4 layers, and arranging the vertical strips in a complementary overlapping checkerboard. Thus, our weave becomes uniformly five layers thick.

4. Tileable Shapes

For any polygonal shape (convex or nonconvex) that tiles the plane, we can generate a weaving of no more than 18 layers, as shown in Figure 4. This method also works for a finite set of polygonal shapes that jointly tile the plane. However, we must make one assumption about the tiling: each tile can appear in only finitely many orientations. Some aperiodic tilings do not have this property [Radin 94, Sadun 98], and our construction does not apply to them.

First, starting from the given tiling, we duplicate it, stacking the two tilings, and then translate one relative to the other so that no tiling edges overlap for positive length. (Tiling edges may overlap at points.) We can find such a translation for any tiling as follows. For each shape in the tiling, define the *minimum feature size* to be the shortest distance between an edge and a non-incident edge; and define the

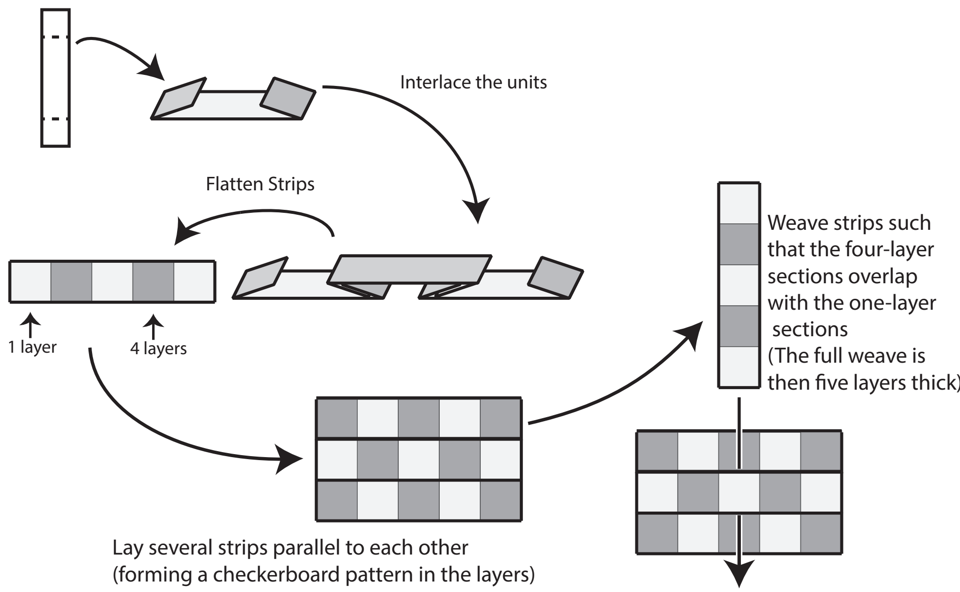


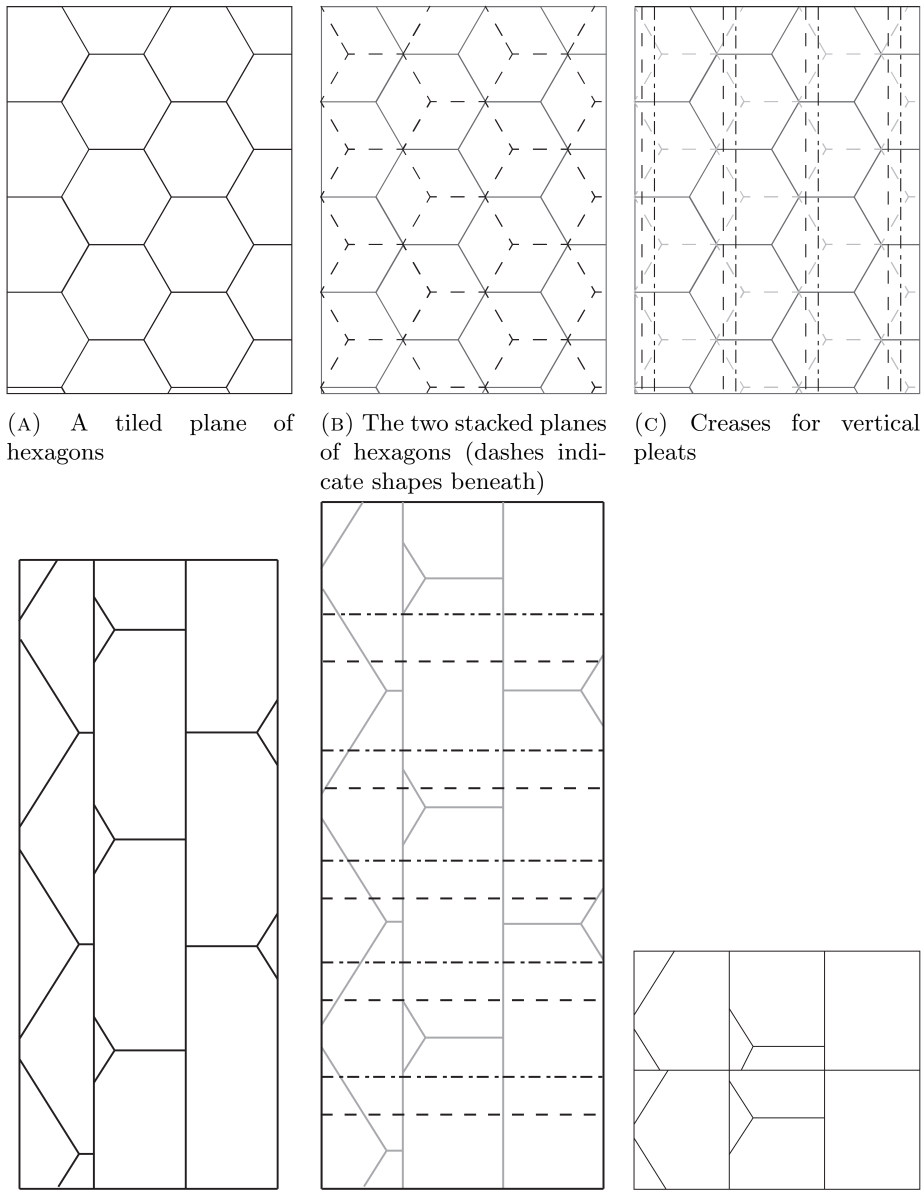
FIGURE 3. A locked infinite weave from a 1×5 rectangle.

minimum feature size of the whole tiling to be the minimum over all tile shapes. (Here we use that the tiling has finitely many tile shapes.) Then we translate by half this minimum feature size, in a direction not parallel to any edge. (Such a direction exists because there are finitely many tile shapes and finitely many edges per tile.)

Now we argue that, for any two edges e, f, the translation e' of e does not overlap f. Because we translate a positive amount not parallel to e, e cannot intersect its own translation e'. Because we translate by less than the minimum feature size, if e and f are not incident, then e' and f cannot intersect, let alone overlap. Finally, if e and f are incident, then they cannot be parallel, so the translation e' can intersect f only at a point.

Next we pleat the composite sheet with many parallel creases in order to lock together all the tiles. For any two overlapping tiles, say untranslated tile s and translated tile t', compute their intersection $s \cap t'$. Because we translated by less than the minimum feature size, the untranslated tiles s, t must be incident or equal. Because there are finitely many tile types, orientations of those tiles, and vertex pairs that could be incident, there are finitely many such tile intersections. For each tile intersection, we measure the projected length in the direction of the pleat, and compute the minimum such length. We uniformly pleat the sheet with two pleats (four creases) per this length, so that independent of shifting, a full pleat hits every tile intersection. The pleating triples the number of layers, for $6 = 2 \cdot 3$ layers thick so far.

Now we prove that these pleats prevent all but a one-dimensional motion, in the direction of the pleat. We have restricted the motion in this way for any pair of overlapping tiles, which forms an infinite bipartite graph of "locking" relations on the tiles; we need to prove that this graph is connected. Consider a vertex v in the original tiling and its translation v', and let T and T' be the set of untranslated and



(A) A tiled plane of hexagons

(B) The two stacked planes of hexagons (dashes indicate shapes beneath)

(C) Creases for vertical pleats

(D) After folding vertical pleats

(E) Creases for horizontal pleats

(F) After folding horizontal pleats. Finished.

FIGURE 4. Weaving an infinite plane of hexagons

translated tiles incident to v and v', respectively. Because we translate by less than the feature size, some translated tile $t' \in T'$ overlaps v, and thus the pleats lock t' together with all tiles $s \in T$. Similarly, some untranslated tile $q \in T$ overlaps v', and thus the pleats lock q together with all $r' \in T'$. In particular, t and t' are locked together, as are q and q', so by transitivity, all tiles in $T \cup T'$ are locked together. Again by transitivity, all tiles in both tilings are locked together.

Finally, we pleat the sheet again, using the same construction as above, but in the orthogonal direction. As a result, all tiles are fully locked together, being unable to move simultaneously in two orthogonal directions. Our completed weave is $18 = 6 \cdot 3$ layers thick.

5. Special Polygon

Using a carefully designed polygon, we can achieve an even smaller number of layers. Specifically, we demonstrate a shape that weaves with only four layers. We start with the folding shown in Figure 5 (left), which has two layers in some places and one in the rest. Locking together that unit with copies of itself produces the weave in Figure 5 (right). However, this weave is not uniformly thick, so we must add additional paper. We add paper until the entire folding is four layers thick, ending up with the crease pattern (and outer polygon) shown in Figure 6. Once we have folded this shape, we lock its corners together with the corners of other identical units to make a four-layer-thick weave.

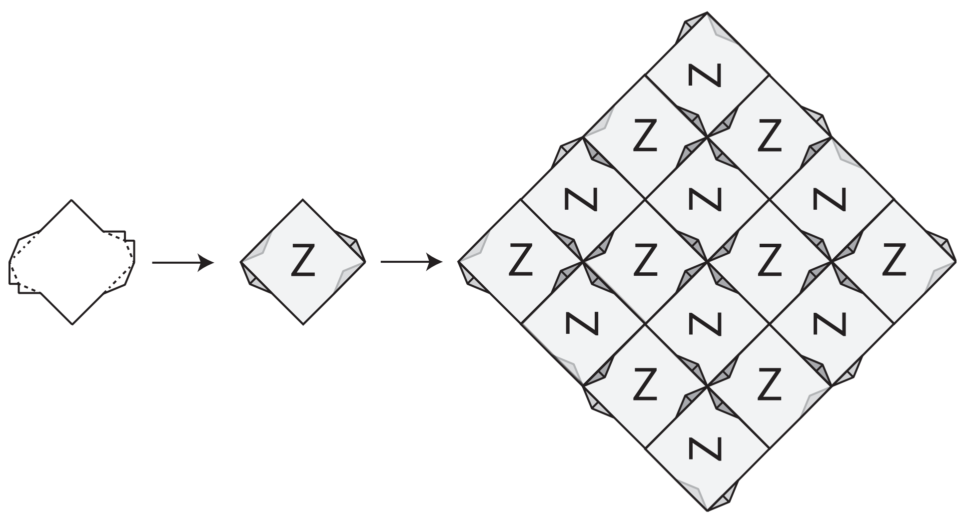


Figure 5. The two-layer unit and resulting woven sheet of irregular thickness (light areas are one layer thick, dark areas are two layers thick). We denote how each piece is rotated with the letter 'Z' (which is 180° rotationally symmetric, like the pieces).

6. Finite Locked Sheets

Our method for making a finite locked sheet from an infinite locked sheet is to treat the tiling as a tube with closed off ends, rather than a sheet. For this approach to work, it must be possible to carve the tiling along tile boundaries into a (roughly rectangular) supertile whose boundary satisfies the topological/metric mating conditions shown in Figure 7a. Matching letters must meet complementarily so that they can be woven together with a uniform number of layers. The idea is that, if we fold the supertile in half vertically to bring the (equally oriented) C's together, then the C boundaries match up as if the tiling continued, and the oppositely oriented

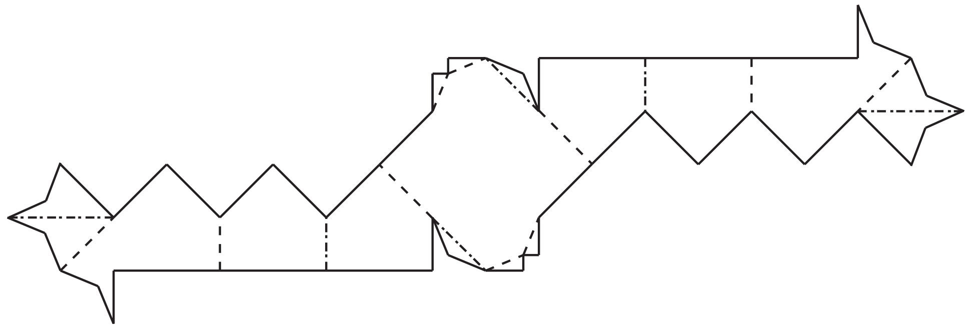
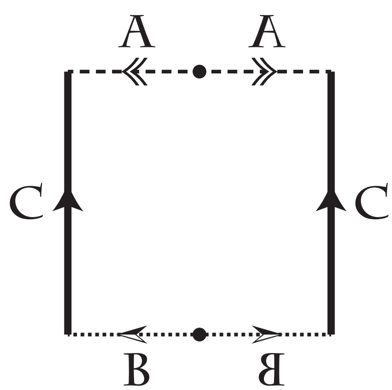
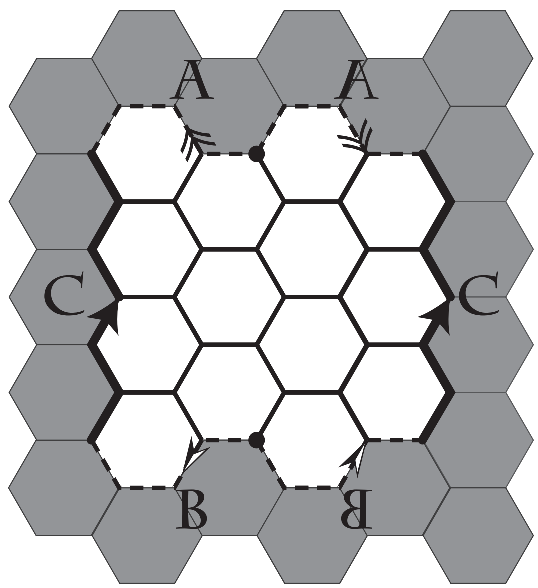


FIGURE 6. The final unit, with extra paper extensions to create a uniformly four-layer-thick weave

A boundaries match up with each other (one reflected by folding), as do the B's. The supertile does not need to tile the plane in a rectangular brick tiling;



(A) Topological disk with boundary labeling of compatibility relations required of the supertile



(B) An example of finding supertiles in a tiling

FIGURE 7. Making finite weaves out of infinite weaves

Figure 7b illustrates the hexagon tiling as an example. The indicated bumpy rectangle is a valid supertile, as is any bumpy rectangle with the same parity. To fold the supertile into a finite woven sheet, first glue together the equal-letter boundaries. Now treat the result as a single (double-covered) tiling with the original tile shapes, and apply the tiling method of Section 4: offset the tiling (on the topological sphere) to form an inner and outer copy, and pleat the two copies together to lock them together. By choosing the pleat widths to evenly divide the supertile, the infinite pleat set becomes a finite pleat set.

Overall, we double the number of layers in the weaving. If we start with rectangles, parallelo-hexagons, or the special polygon, we can use the special weavings from Sections 2, 3, and 5, instead of the general tiling weaving.

7. Dollar Bill Folding

One application of this sheet weaving—and the original reason we explored this project—is to create a woven locked sheet of dollar bills. Glass blowers use sheets of newsprint folded into a "hot pad" to effectively touch hot glass without burning themselves. As part of an art project, we thought that it would be interesting to use a sheet of woven dollar bills instead of newsprint to shape the glass. The burnt dollar bills would then be displayed alongside the glass as a single sculpture, entitled "Money to Burn".

It is important for glass-blowing hot pads to be uniformly and appropriately thick: a too-thin pad will burn the user, while a too-thick pad will mask too much feeling from the user. We based our original design off of Jed Ela's MoneyWallet [Ela 04]. However, his wallet is not uniformly thick, so we created our own design.

The conventional children's weave is too thin to use as a pad and slips apart easily, so we developed the precursor to the eight-layer weave of Section 2. Because we needed the pad to be thick to handle the heat of the glass, we created a sixteen-layer locked weave, as follows: fold the two narrower edges of the rectangle to the middle, as in Figure 8. Then, fold the longer edges to the middle, over the other edges. Unfold the longer edges, and interlace units together into a long strip. Fold all the longer edges together into the middle along the strip, creating a long locked strip. Create several of these strips, and weave them together into a sheet, as in Figure 9. This whole method is the same as that of the general rectangle weave in Figure 1, except with strips that are twice as thick. Each strip is eight layers thick, which, when woven, makes 16 layers. Using the tube method of Section 6, we created a finite locked sheet of 32 layers.

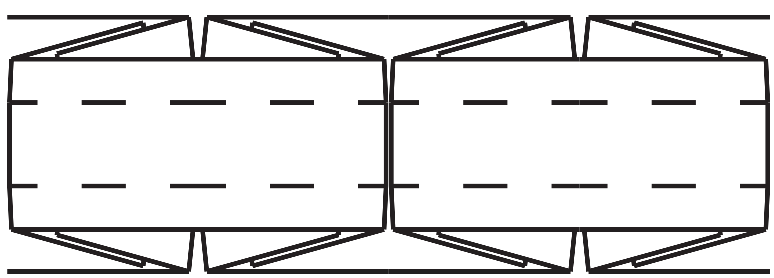


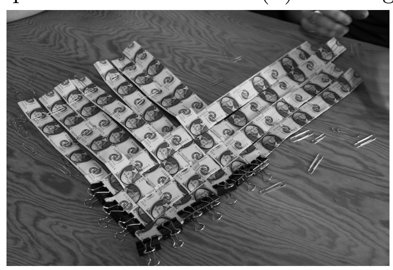
Figure 8. Creating the dollar bill strip, as seen from the top (mountain fold along red lines to lock)

However, any weave has small gaps between strips. In the mathematical model, these gaps are dismissed as "infinitesimally small" points, but in physical models, the thickness of the paper may cause tiny but noticeable gaps. For glass blowing, these holes let hot steam pass through, burning the user. We therefore lined the inside of the final weave with eight additional layers of dollar bills (roughly 24 to 30 dollars' worth).

Our final dollar bill pad, shown in Figure 10, is forty layers thick, and contains 140 to 146 dollar bills.

(A) Creating a strip of dollar bills

(B) Weaving the first strip



(C) The partial sheet

FIGURE 9. Weaving strips of dollar bills together

FIGURE 10. The finished woven sheet of dollar bills

8. CONCLUSION

An intriguing open question is whether there is a general method for creating a locked infinite (or even finite) sheet from any possible shape.

In a forthcoming paper with Barry Hayes, we prove that a two-layer weave is possible, for an appropriately shape of paper.

Acknowledgments

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References

[de Villiers 12] Michael de Villiers. "Relations between the sides and diagonals of a set of hexagons." *The Mathematical Gazette* 96 (July 2012), 309–315.

[Ela 04] Jed Ela. "MoneyWallet." Moneywallet.org, 2004. Accessed 12 June 2012.

[Radin 94] Charles Radin. "The Pinwheel Tilings of the Plane." *Annals of Mathematics* 139:3 (1994), 661–702.

[Sadun 98] Lorenzo Sadun. "Some Generalizations of the Pinwheel Tiling." *Discrete & Computational Geometry* 20:1 (1998), 79–110.

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