Matching Points with Things

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Representing a matching between pairs of planar objects as a set of non-crossing line segments is a natural problem in computational geometry. It is well known, for instance, that given two sets of n points in the plane, say n red points and n blue points, there always exists such a non-crossing matching between red and blue points. In particular, it is not difficult to show that the minimum Euclidean length matching is non-crossing. Kaneko and Kano [3] survey a number of related results.

We investigate related questions for general planar objects instead of points. In this case, the matching is represented by line segments, the endpoints of which belong to the corresponding matched objects. Note that, from the above result, a non-crossing matching always exists between two sets of objects. However, in this paper, we consider the problem of finding a matching when we are given object pairs as input. Since pairs are enforced, the existence of a non-crossing matching is no longer guaranteed.

The problem can be seen as a 1-regular graph drawing problem with constraints on the location of the vertices. In all variants that we consider, every element p_i in one of the sets of objects is a single point. In the second set, the objects t_i can be lines, segments or point sets. More precisely, let $P := \{p_i\}_{i=1}^n$, and $T := \{t_i\}_{i=1}^n$. A *matching* for (P, T) consists of a set of line segments (or *edges*) of the form $\{p_i m_i\}_{i=1}^n$, where $m_i \in t_i$. A matching is *non-crossing* if no pair of its edges properly cross. This is illustrated in Figure 1.

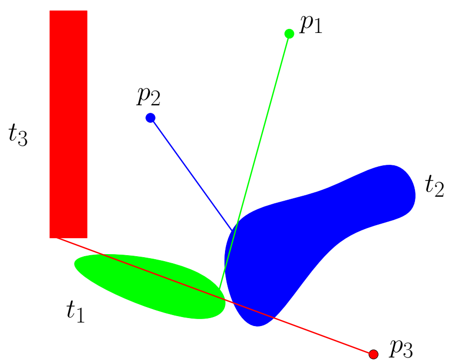

Figure 1: Example of a non-crossing matching for a set $P = \{p_1, p_2, p_3\}$ of points and a set $T = \{t_1, t_2, t_3\}$ of planar objects.

We consider the problem of deciding whether a non-crossing matching exists for a given pair (P, T). We also consider the problem of finding one that minimizes either the maximum length of an edge, or the sum of the lengths.

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Related work
Problems on matchings have an important role in combinatorial Graph Theory, for both theoretical and applied aspects. A large body of research has been devoted to this, (e.g., see the book by Lovasz and Plummer [4]). The family of geometric problems that we consider is related to problems in shape matching, colour-based image retrieval, music score matching and computational biology.

Suppose that the vertices of a graph are points in the plane, edges are rectilinear segments, and edge weights are Euclidean distances. It is a simple fact that the sum of any pair of opposite sides of a convex quadrilateral is strictly smaller than the sum of the diagonals. Remarkably this implies than the minimum weight matching in any realization of the complete graphs K_{2n} and $K_{n,n}$, will consist of pairwise non-crossing segments.

These problems can be solved using the generic algorithms for weighted graphs. However, in the planar case just mentioned, Vaidya [5] proved that it is possible to obtain specific algorithms with better running times. This was later improved to $O(n^{2+\varepsilon})$ by Agarwal et al. [1]. Similar results have been obtained for other variations such as *bottleneck matching* or *uniform matching*, in the work of Efrat, Itai and Katz [2].

Our results
We study the case where the objects t_i are finite point sets. We prove that the decision problem is NP-complete if the t_i have size greater than 2; the proof is a reduction from 3-SAT. When all t_i are pairs of points, we can decide in quadratic time.

When the t_i are line segments, the decision problem is NP-complete, even if the segments have a fixed number of orientations, or if they all have unit length. On the other hand, we consider various special cases, for which polynomial-time algorithms are provided:

- the segments are edges of a convex polygon containing all p_i
- the segments belong to a single line and are disjoint
- the segments belong to a single line, but are not necessarily disjoint

In fact, for the first two special cases, the minimum-sum of edge lengths can also be found in polynomial time.

Finally, we consider the problem of matching points with lines. In this case, a non-crossing matching always exists, but the minimization problems are NP-hard. However, if the number of distinct directions is k, then a solution can be found, approximating the optimal by a factor of $1/\sin(\frac{\pi}{2k})$.

References

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