Interlocked Open and Closed Linkages with Few Joints

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Abstract

We study collections of linkages in 3-space that are *interlocked* in the sense that the linkages cannot be separated without one bar crossing through another. We explore pairs of linkages, one open chain and one closed chain, each with a small number of joints, and determine which can be interlocked. In particular, we show that a triangle and an open 4-chain can interlock, a quadrilateral and an open 3-chain can interlock, but a triangle and an open 3-chain cannot interlock.

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1 Introduction

Consider a simple polygonal chain, either an open *arc* or a closed *polygon*, that is embedded in 3-space. We view the vertices of the chain (except the endpoints of an open chain) as universal *joints*, and the edges of the chain as rigid *bars*. We call a chain with k bars a k*-chain*. A *motion* of the chain is a motion of the vertices that preserves the length of the bars, and never causes bars to cross. In particular, a *straightening of an open chain* is a motion that makes all joint angles become 180°. We say that a collection of disjoint, simple chains can be *separated* if, for any distance d, there is a motion whose result is that every pair of points on different chains has distance at least d. If a collection cannot be separated, we say that its chains are *interlocked*. If a single chain cannot be straightened, we say that it is *locked*.

It is known that a single, open chain in 3-space, having as few as 5 bars, can be locked [1,2]. Other classes of chains are known to be unlocked, but the complexity of deciding whether a given chain can be unlocked is not known. One decision procedure applies the roadmap algorithm for general motion planning [3,4], which runs in exponential time.

Our work is inspired by a question posed by Anna Lubiw [5]: Into how many pieces must a chain be cut so that the pieces can be separated and straightened? This problem is motivated by protein molecules, which can be modeled by polygonal chains, and, according to some theories, temporarily split apart in order to reach the minimum-energy folding.

We can observe easy upper and lower bounds for Lubiw's problem: some n-chains require cutting at least $\lfloor (n-1)/4 \rfloor$ vertices for separation, and no chain requires cutting of more than $\lfloor (n-1)/2 \rfloor$ vertices. The lower bound is obtained by concatenating many copies of the 5-bar "knitting needles" example from [1,2], each sharing one bar with the next as in Fig. 1. Observe that each copy of the locked 5-bar chain must have one of its four interior vertices cut. The upper bound is obtained by cutting every second joint of a chain, and observing

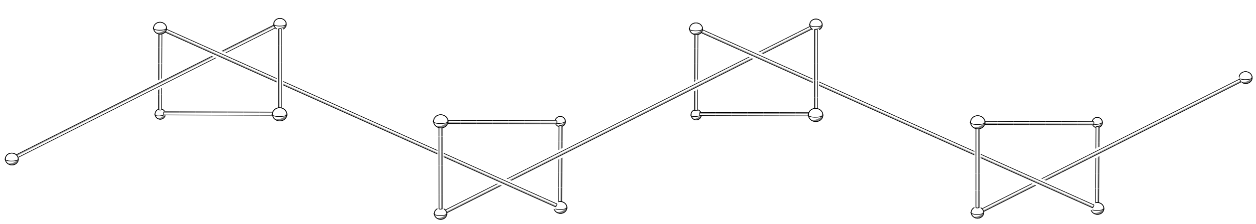


Fig. 1. An $n = 17$ bar chain that requires cutting at least $\lfloor (n-1)/4 \rfloor = 4$ vertices to separate.

that the resulting 2-bar pieces ("hairpins") can be rigidly separated arbitrarily far by dilating from a point, because the pieces are starshaped sets. This separation motion dates back at least to de Bruijn in 1954 [6], where he used it to prove separability of convex objects; the same motion was shown to apply to the more general situation of starshaped objects by Dawson in 1984 [7], and the algorithmic side of this result is described by Toussaint in 1985 [8]. See also [9].

While Lubiw's problem motivated our original interest in interlocked open chains, we explore here interlocking for combinations of open and closed chains. In the next section, we resolve how many bars are needed by each chain in order to obtain an interlocked pair, as summarized in Table 1.

Sec	**Chain 1**		**Chain 2**		**Result**
2	closed	triangle	open	3-chain	Cannot Interlock
3.1	closed	triangle	open	4-chain	Can Interlock
3.2	closed	quadrilateral	open	3-chain	Can Interlock

Table 1
Our results on when an open chain and a closed chain can interlock. A claim that a k-chain can interlock holds also for any l-chain with $l > k$, and a claim that a k-chain cannot interlock holds also for any l-chain with $l < k$.

2 Triangle and 3-chain Cannot Interlock

We begin by showing that a triangle and a 3-chain cannot interlock. As we will see later, this is in some sense a maximal non-interlocking configuration.

Theorem 1 *An open 3-chain cannot interlock with a triangle.*

PROOF. We follow this notation: $\triangle abc$ lies in plane H, and the 3-chain C has vertices (p_0, p_1, p_2, p_3) and bars (l_0, l_1, l_2). First assume C is not planar; otherwise, make C nonplanar by a small motion. Let L_i be the support line of l_i and define points $q_i = L_i \cap H$.

(1) Bar l_1 intersects the closed $\triangle abc$. In this case, it is possible to move bar l_0 and bar l_2 within the plane that it forms with l_1 so that the angle at the joint shared with l_1 is arbitrarily close to either 0 or π, because one of the two wedges spanned by these two motions does not intersect any other edge. Once both end bars have been moved to that position, C is arbitrarily close to a single bar which can be translated in the direction $\overrightarrow{p_1p_2}$.
(2) Bar l_1 does not intersect the closed $\triangle abc$. Because configuration C is non-self-intersecting, we can assume that the points $\{q_0, p_1, p_2, q_2\}$ do not lie on a common plane, or equivalently $\{q_0, q_1, q_2\}$ are not collinear. Denote the line containing q_0 and q_2 by $Q_{0,2}$, as in Fig. 2. In fact, for any position of l_1 such that $(L_1 \cap H) \notin Q_{0,2}$, the lines containing q_0p_1 and p_2q_2 do not intersect, and do not intersect the edges of $\triangle abc$. Thus the motion that translates l_1 in a direction orthogonal to $Q_{0,2}$ and parallel to H, away from $\triangle abc$, while maintaining L_0 and L_2 through the original points q_0 and q_2, will avoid self-intersection.[3]

[3] See `http://www.cs.smith.edu/~orourke/Interlocked/` for an animation of this motion.

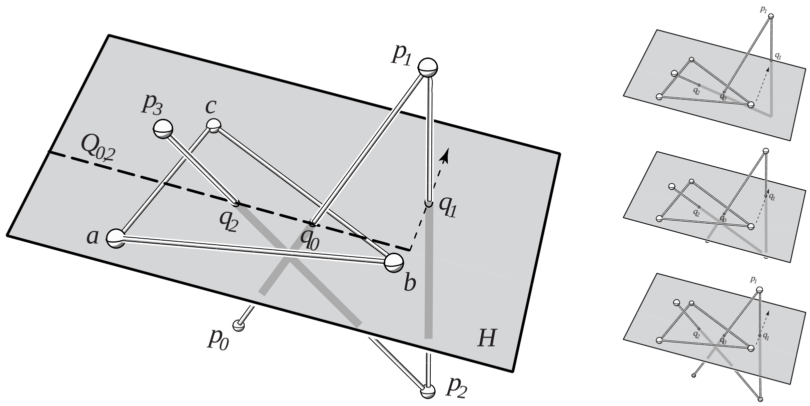


Fig. 2. Translate l_1 so that the point $q_1 = L_1 \cap H$ moves away from $Q_{0,2}$. Keep the points q_0 and q_2 fixed in H, so that the lines L_0 and L_2 pivot about $q_0 = L_0 \cap H$ and $q_2 = L_2 \cap H$ as l_1 moves. This separates the 3-chain from $\triangle abc$.

3 Interlocked Examples and the Topological Method

Our two proofs that chains are interlocked follow a similar structure in what we call the *topological method*. We imagine tying the two ends of the open chain with a long rope near infinity, which defines a topological *link* (multicomponent knot) [10, p. 17]. For the two chains to separate, they must form the trivial link (referred to as 0^2_1; see later). First we show that before this happens, the ends of the open chain must get close to the closed chain. Second we argue that this proximity is impossible before changing the topology of the link. Finally we prove that this circularity leads to a contradiction, so the chains are interlocked.

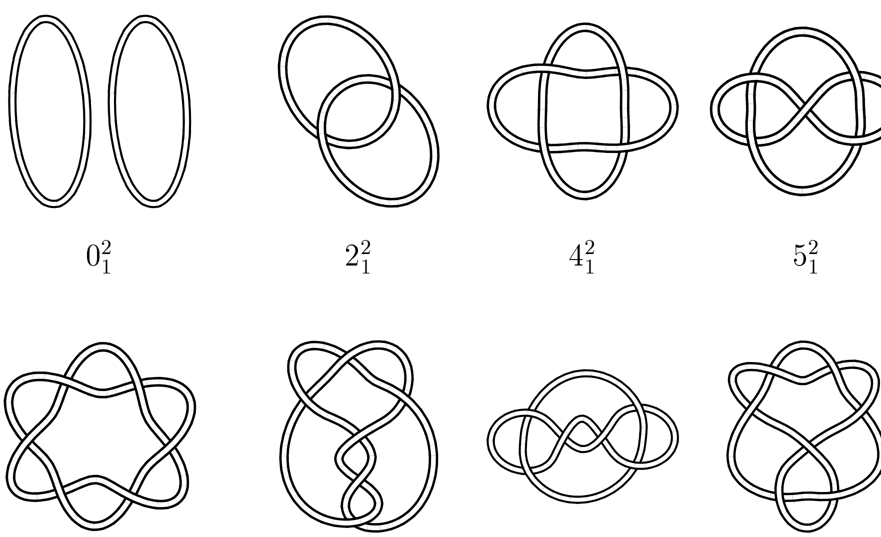


Fig. 3. The first few two-component links.

To make connections to known mathematics for links, we will refer to some links by their numbers from standard tables. See [10, p. 287] or [11, p. 1086]. Tables of links are often organized by (minimum) crossing number. The superscript in the link notation is the number of components, for us always 2. The subscript is an arbitrary table index. See Fig. 3 [4].

3.1 *Triangle and 4-chain*

We begin with the configuration illustrated in Fig. 4.

Theorem 2 *A triangle can interlock with a 4-chain.*

PROOF. We choose the following notation for the configuration of Fig. 4: A triangle abc lies in a plane H, with H^+ the halfspace above and H^- the halfspace below H. Let the circumcircle of $\triangle abc$ have center o, and radius r.

The 4-chain alternates points and bars p_0, l_0, p_1, l_1, ..., l_3, p_4 with the following placements: p_0 is in H^-, bar l_0 crosses the interior of $\triangle abc$, and ends at a point p_1 above o. Bar l_1 crosses the interior of $\triangle abc$ again, so $p_2 \in H^-$. Bar l_2 crosses H outside of $\triangle abc$, and l_3 crosses the wedge formed by l_0 and l_1 above H. So $\{p_0, p_2\} \subset H^-$ and $\{p_1, p_3, p_4\} \subset H^+$.

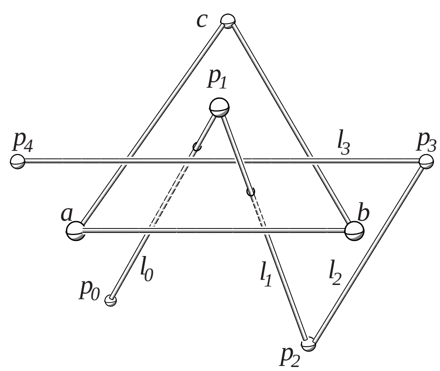


Fig. 4. A triangle and a 4-chain can lock.

Let R be the real number $r + |l_1| + |l_2|$, and set the length of l_0 and l_3 to $20R$. Consider the open ball B of radius $15R$, and the ball B' of radius $4R$, both centered at o. Initially, p_0 and p_4 lie outside of B, while a, b, c, p_1, p_2 and p_3 all lie inside $B' \subset B$. As long as p_0 and p_4 stay outside B and all other vertices stay inside B, we can attach a sufficiently long unknotted string between p_0 and p_4 that remains outside B, and thus is never crossed by any of the bars, and our configuration is equivalent to the link 5^2_1. The non-interlocked configuration corresponds to two separable unknots 0^2_1, so any motion separating this configuration would require p_0 or p_4 to enter the ball B or p_1, p_2, or p_3 to leave B.

Consider the first event when any p_i, $i = 0, \ldots, 4$ touches the boundary of B. Then before or at that event, points p_1, p_2 and p_3 must be out of B' but still inside B: When p_0 touches

[4] link images produced by Robert Scharein's `knotplot` program `http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html`.

B, point p_1 must be exterior to B' by at least R, and therefore p_2 and p_3 are also exterior to B'. See Fig. 5. The same applies for when p_4 touches the boundary of B. When any one of p_1, p_2 or p_3 touches the boundary, the other two are at least at a distance $14R$ from o and so are outside of B'. Since we consider the first such event, there must be an instant before that when all three points are outside B' but still inside B.

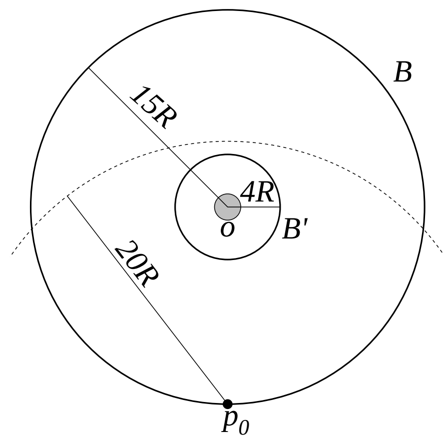


Fig. 5. When p_0 touches B, point p_1, p_2 and p_3 must be exterior to B'.

At this time, the only elements possibly inside B', besides $\triangle abc$, are the two bars l_0 and l_3. Then either one of l_0 and l_3 crosses the interior of $\triangle abc$, or both do, or neither do. The first case corresponds to a link 2_1^2 and the third case to two separable unknots 0_1^2; neither of these are equivalent to our starting configuration (in the knot theoretical sense). Since the rope and the bars have not crossed, the topology of the configuration cannot have changed and so these cases lead to a contradiction.

The case in which both l_0 and l_3 cross $\triangle abc$ requires a careful analysis. Because end vertices p_0 and p_4 are still outside of the open ball B, we can replace the string joining them by a great arc γ on the boundary of B. Let T be the plane parallel to l_0 and l_3, and passing through o. Consider the orthogonal projection of the 4-bar linkage onto T. Note that in the projection, the lengths of bars l_0 and l_3 are preserved, and all other segment lengths are at most their original lengths. Let q_0 be the intersection of l_0 and plane H. The triangle $\triangle abc$ is contained in a ball of radius $2R$ centered at q_0, and joints p_1, p_2 and p_3 lie in a ball of radius R centered at p_1. Since p_1 is outside B' and q_0 is inside the circumcircle of $\triangle abc$, the distance between those two points is larger than $3R$, and that distance is preserved in the projection. Thus, the projections of the two balls are disjoint and we can separate the projections of p_1, p_2 and p_3 from the projections of p_0, p_4 and $\triangle abc$ by a line (This separation is necessary to exclude cases such as the one shown in Fig. 6.), and the two bars l_1 and l_2 can be replaced by a single bar joining p_1 and p_3 without changing the topology of the link. By enumerating all possible above/below combinations for the crossings in that projection, we can infer that configuration is equivalent to 0_1^2, which is two separated, unknotted links, or to 4_1^2, which is shown in Fig. 7. But neither of these are topologically equivalent to our starting configuration, so this first event could never happen.

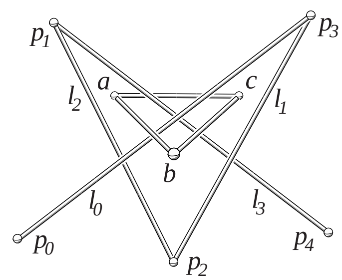


Fig. 6. This configuration is incompatible with the fact that p_0 or p_4 touches the boundary of B.

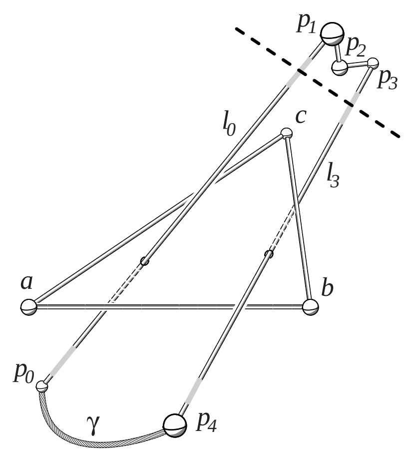
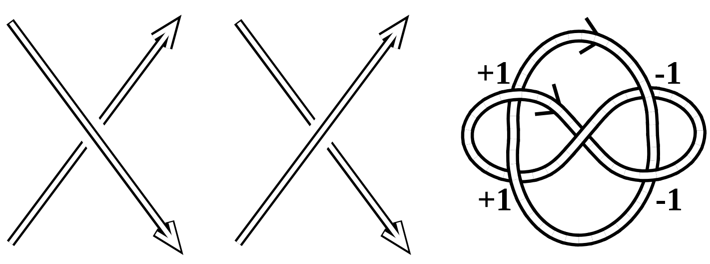


Fig. 7. The link 4_1^2, formed when bars l_0 and l_3 both pass through the interior of $\triangle abc$. (Not to scale; gray segments indicate omissions.) Joints $\{p_1, p_2, p_3\}$ can be separated from $\{a, b, c, p_0, p_4\}$.

Note that a similar argument can be used to show that the chains in Fig. 6 are interlocked as well.

3.2 Quadrilateral and 3-chain

In the following, we will use what is known as the *linking number* of a two component link. We first arbitrarily orient both components of the link. Then each crossing drawn in the projection of the link has one of two types, associated with a value $+1$ or -1. See Fig. 8.



$+1$ $\quad -1 \quad$ $Link(5_1^2) = 0$

Fig. 8. Sign of a crossing.

The *linking number* of the link is half the sum of the values of all crossings between the different components; crossings of a component with itself are not counted. For example, the link 5^2_1 has 5 crossings, but only four of them involve both components. The sum of the values of the four crossings is 0, which yields a linking number of 0. Note that if the orientation of one of the components is reversed, then the linking number is negated. It can be proved using some elementary knot theory that the linking number of an oriented link is an *invariant*, that is, it has the same value for all drawings of the oriented link [10, p. 21].

Theorem 3 *A 4-gon can interlock with a 3-chain.*

PROOF. Let the 4-gon be *abcd*, and again use (l_0, l_1, l_2) and (p_0, p_1, p_2, p_3) to represent the bars and vertices of the 3-chain. Starting with the configuration of Fig. 9, let $R = |ab| + |bc| + |cd| + |l_1|$ and set the length of l_0 and l_2 to $20R$. Consider the open ball B of radius $15R$, the ball B' of radius $4R$, and the ball B'' of radius R, all three centered at a. As in the previous proof, we connect p_0 to p_3 by a string exterior to B. The resulting link is now 6^2_1. We again argue that in order to separate the 4-gon from the 3-chain, p_0 or p_3 has to enter the ball B or p_1 or p_2 have to leave B. Before that, there must be an instant when p_0 and p_3 are still outside B, p_1 and p_2 are still inside B but out of B', and the only elements possibly inside B', besides *abcd*, are the two edges l_0 and l_2.

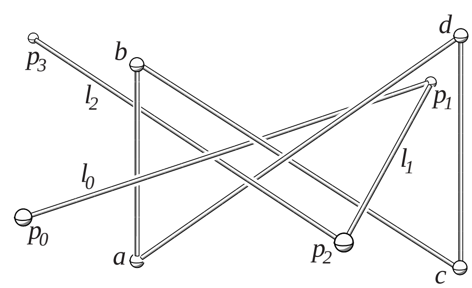


Fig. 9. A quadrilateral and a 3-chain can lock.

If neither l_0 nor l_2 intersects B'', then the configuration is the link 0^2_1, contradicting that the topology cannot have changed. If one of the two end bars, say l_0, intersects B'', let q_0 be a point of $l_0 \cap B''$. We project the configuration onto a plane parallel to l_0 and l_2, preserving the distances along those two bars. As in the previous proof, because the length of the segment $q_0 p_1$ is preserved in the projection, only the interiors of l_0 and l_2 can intersect the projection of B''. This implies that the linking number of the configuration will be the sum of the values induced by l_0 and *abcd*, and the values induced by l_2 and *abcd*, divided by 2. Notice that the total of the values induced by a straight edge and a 4-gon is at most 2, and so the linking number of the configuration is at most $(2+2)/2 = 2$. But the linking number of 6^2_1 is 3. Because the linking number is an invariant, the topology of the configuration must have changed, a contradiction.

4 Open Problems

Many open problems remain in the context of interlocking pairs of open chains, which have close connections to the motivating problem of Lubiw. For each value of i, what is the smallest j for which an i-chain can interlock with a j-chain?

The topological method of Theorems 2 and 3, where we used a "rope" to close one open chain to form a topological linkage, does not easily extend to pairs of open chains. Two ropes would be needed, and their potential interactions would need to be controlled. To extend this work, therefore, we will be investigating a geometric method that establishes a collection of geometric facts and shows that there can be no first violation. We believe that we can use such a method to establish three conjectures: that a 3-chain can interlock with a 4-chain, that three 3-chains can interlock, but that two 3-chains cannot interlock even in the presence of any finite number of 2-chains.

The proof of Theorem 3 depends upon a tetrahedron formed by the 4-gon, and does not show that a 3-chain and a k-gon can interlock for any $k > 4$. In fact, adding any small edge to the 4-gon would allow the 3-chain to escape. On the other hand, our conjecture that a 3-chain can interlock with a 4-chain, once established, would imply that a 3-chain can interlock with a k-gon for any $k \geq 5$ by connecting the endpoints of the 4-chain with one or more edges.

Chains that model physical objects, such as robot arms or protein backbones, often have restrictions placed on the motion of a joint. There are a number of interesting problems for open and closed chains under various restrictions on motions. For example, we conjecture that a rigid, open 3-chain can interlock with a flexible, open 3-chain.

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References

[1] J. Cantarella, H. Johnston, Nontrivial embeddings of polygonal intervals and unknots in 3-space, J. Knot Theory Ramifications 7 (8) (1998) 1027–1039.

[2] T. Biedl, E. Demaine, M. Demaine, S. Lazard, A. Lubiw, J. O'Rourke, M. Overmars, S. Robbins, I. Streinu, G. Toussaint, S. Whitesides, Locked and unlocked polygonal chains in 3D, in: Proc. 10th ACM-SIAM Sympos. Discrete Algorithms, 1999, pp. 866–867, full version LANL arXive cs.CG/9910009.

[3] J. Canny, A new algebraic method for robot motion planning and real geometry, in: Proceedings of the 28th Annual Symposium on Foundations of Computer Science, Los Angeles, California, 1987, pp. 39–48.

[4] J. Canny, Some algebraic and geometric computations in PSPACE, in: Proceedings of the 20th Annual ACM Symposium on Theory of Computing, Chicago, Illinois, 1988, pp. 460–469. URL `http://www.acm.org/pubs/citations/proceedings/stoc/6221%2/p460-canny/`

[5] E. D. Demaine, J. O'Rourke, Open problems from CCCG'99, in: Proc. 12th Canad. Conf. Comput. Geom., 2000, pp. 269–272. URL `http://cs.smith.edu/~orourke/ShortestPaths/`

[6] N. G. de Bruijn, Nieuw Archief voor Wiskunde 2 (1954) 67, problems 17 and 18. Answers in *Wiskundige Opgaven met de oplossingen*, 20:19–20, 1955.

[7] R. J. Dawson, On removing a ball without disturbing the others, Math. Mag. 57 (1984) 27–30.

[8] G. T. Toussaint, Movable separability of sets, in: G. T. Toussaint (Ed.), Computational Geometry, North-Holland, Amsterdam, Netherlands, 1985, pp. 335–375.

[9] J. Snoeyink, J. Stolfi, Objects that cannot be taken apart with two hands, Discrete Comput. Geom. 12 (1994) 367–384.

[10] C. C. Adams, The Knot Book, W. H. Freeman, New York, 1994.

[11] E. W. Weisstein, CRC Concise Encylopedia of Mathematics, CRC Press LLC, 1999.