How Efficiently Can Nets of Polycubes Pack a Rectangle?

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1 Introduction

We study a problem of packing nets of polycubes into a rectangle area. More specifically, given an $m \times n$ rectangle board with grid lines, we pack arbitrary nets of a specified polycube (a unit cube, for example) aligned with the grid lines, and we are interested in packing as many nets as possible in the board. Since the problem setting is quite simple, fun and could have potential practical applications, it has attracted a lot of interest and attention since early times [1], however, very few results can be seen formally. Recently, Inoha et al. [2] studied it systematically. They checked the correctness of several facts known as folklore (e.g., [4]) and improved them. In this research we push it into the same direction, and show several new results in the case of nets of cubes. We also consider the case of bicubes.

2 Cubes

It is well-known that when we develop a cube along edges (edge unfolding), it has eleven distinct nets (up to mirror and rotation symmetry). They form a part of 35 hexominoes. A primitive question is if nets of the cube can exactly pack a rectangle, but the following fact is also known.

Theorem 0 *Any rectangle cannot be exactly packed by nets of the cube.*

This is proven by small case analyses by considering which net may cover a corner of a rectangle [2, 3].

Then the next interest goes to how efficiently can nets of the cube pack a rectangle, that is, we like to minimize the number of uncovered cells in a packing. The exact minimum values are obtained for rectangles of small sizes as shown in Table 1. This table gives several interesting observations, and strongly motivates to investigate these values in general. We try to derive both upper and lower bounds theoretically.

Table 1: The minimum number of uncovered cells in $m \times n$ rectangles for small m and n (all values are proven to be optimum by some means). These are the largest known sizes to the best of our knowledge.

2	4			
3	6	9		
4	8	6	4	
5	4	9	8	7
×	2	3	4	5

6	6	6	6	6	12								
7	8	9	4	11	6	7							
8	4	6	8	10	6	8	10						
9	6	9	6	9	6	9	6	9					
10	8	6	4	8	6	10	8	6	10				
11	4	9	8	7	6	11	10	9	8	7			
12	6	6	6	6	12	6	6	6	6	6	6		
13	8	9	4	11	6	7	8	9	10		6	7	
14	4	6	8	10	6	8	10	6	8		6	8	
×	2	3	4	5	6	7	8	9	10	11	12	13	14

As for upper bounds, we can show the following.

Theorem 1 *Let $m, n \in \mathbb{N}$. If $m \geq n \geq 2$, then nets of the cube can pack an $m \times n$ rectangle leaving at most 14 uncovered cells.*

Proof sketch. The proof is done by case analyses, firstly by the board sizes and then by the way similar to the proof of Theorem 0. We assume without loss of generality that $m \geq n$ for the board of size $m \times n$.

For $m \times n$ with $m \geq n \geq 6$ and $m \geq 8$, we have the claim by detailed construction of packing patterns. Verification using BurrTools [5] proves that rectangles of dimensions $m \times n$ with $m \geq n \geq 6$ and $m \in \{6, 7\}$ can be packed with cube nets leaving at most 12 cells uncovered.

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Thus all that remains is to consider rectangles with dimensions $m \geq n$ with $5 \geq n$. The cases where $5 \geq m$ are listed in Table 1 (left), and we can confirm that solutions leaving at most 9 cells uncovered are possible for all such dimensions. Packings for each subcase of $m \geq 6$ are considered separately. In each subcase, we can pack nets in such a way that at most 5 and 7 cells are uncovered at the right and left end of the rectangle, respectively, for a total of at most $12 < 14$ cells uncovered. □

We next consider lower bounds on the number of uncovered cells.

Theorem 2 *Let $m, n \in \mathbb{N}$. If $\{m, n\} \notin \{\{1,1\},\{1,2\},\{1,3\}\}$, then any packing of nets of the cube into a $m \times n$ rectangle leaves at least 4 uncovered cells.*

Proof sketch. The proof is done again by case analyses by the board sizes.

For $m \times n$ with $m \geq n \geq 2$ and $m \geq 6$, we have the claim by carefully constructing such packing patterns. We here remark that the value 4 is incurred by four corners of a rectangle, that is, packing a corner by a net leaves at least one uncovered cell and no net can cover two corners simultaneously for sufficiently large boards.

For $m \times n$ with $5 \geq m \geq n \geq 2$, verification using BurrTools proves that $m \times n$ rectangles can be packed with cube nets leaving at least 4 cells uncovered, as we can see in Table 1 (left). □

3 Bicubes

Similarly to the cube, we obtained the following facts by exhaustive enumerations with computer programs.

Fact. There is a unique bicube ($1 \times 1 \times 2$), and it has 723 distinct nets. There are two tricubes ($1 \times 1 \times 3$ and L-shape), and the $1 \times 1 \times 3$ tricube has 15087 distinct nets.

The first question is if nets of the bicube can exactly pack a rectangle, and surprisingly we can do it.

Theorem 3 *There is a rectangle that can be exactly packed by nets of the bicube.*

Proof. The proof is by demonstration, seen in Figure 1. The surface area of the dicube is 10, and here a 20×26 rectangle is exactly packed by using 52 nets, which come from 11 distinct (up to symmetry) nets. □

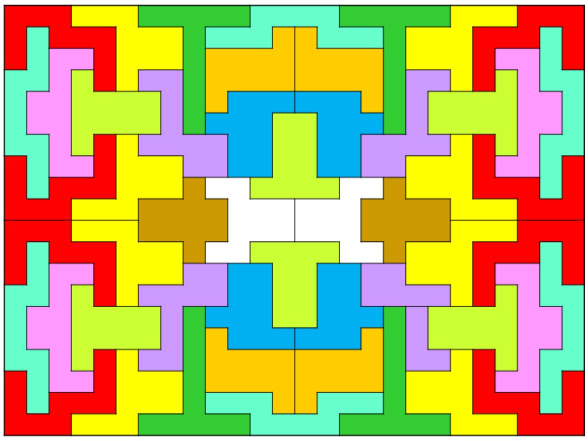

Figure 1: A 20×26 rectangle exactly packed by nets of the bicube.

4 Discussion

We discuss about the cube case. Odawara [4] gave a following observation.

Observation. For boards of sizes 7×7 or larger, the number of uncovered cells ranges from 6 to 11.

It turns out that we proved that it ranges from 4 to 14 for boards of any size $m \times n$ of $m \geq n \geq 2$, which is stronger results in some sense, while the values of 6 and 11 of the range in the observation are still effective. We pose the following new conjecture.

Conjecture. For $6 \times 6k$ board, the number of uncovered cells is 12 (exactly $6k - 2$ nets are packed).

References

[1] S. W. Golomb. *Polyominoes (2nd edition)*. Princeton University Press, 1994.

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