# Folding Small Polyominoes into a Unit Cube

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## Abstract

We demonstrate that a $3 \times 3$ square can fold into a unit cube using horizontal, vertical, and diagonal creases on the $6 \times 6$ half-grid. Together with previous results, this result implies that all tree-shaped polyominoes with at least nine squares fold into a unit cube. We also make partial progress on the analogous problem for septominoes and octominoes by showing a half-grid folding of the U septomino and $2 \times 4$ rectangle into a unit cube.

## 1 Introduction

Which polyominoes fold into a unit cube? Aichholzer et al. [ABD$^+$18] introduced this problem at CCCG 2015, along with a variety of different models for folding. Table 1 summarizes the main models and known results. We focus here on the powerful ***half-grid model*** (the bottom two rows of Table 1) where

1. the polyomino can be folded along horizontal, vertical, and $\pm 45°$ diagonal creases;
2. every crease has endpoints whose coordinates are integer multiples of $\frac{1}{2}$; and
3. each crease can be folded by $\pm 90°$ or $\pm 180°$.

In particular, the paper can overlap itself, using multiple layers to cover the cube (as in origami, but unlike polyhedron unfolding), so long as the paper covers every point of the cube. Look ahead to Figures 4, 5, and 6 for examples of foldings in the half-grid model.

A strong positive result [ABD$^+$18, Theorem 3] is that every polyomino of at least ten squares can fold into a unit cube in the half-grid model.[1] In this paper, we tackle the analogous problem for smaller polyominoes, with at most nine squares.[2]

Any polyomino folding into a unit cube has at least six squares (because the cube has surface area 6). Indeed, for hexominoes, the half-grid and diagonal features of the model are not useful, because the folding cannot have any overlap (again by an area argument). Therefore, the hexominoes that fold into a unit cube are exactly the eleven hexomino nets of the cube; see e.g. Gardner [Gar89] for the list. Aichholzer et al. [ABD$^+$18, Fig. 16] verified by exhaustive search that this claim remains true even if we allow cutting the polyomino with slits until the dual graph (with a vertex for each square and edges for uncut edge adjacency) is a tree; we call these ***tree-shaped polyominoes***.

In between this solved hexomino case and the universally foldable $\geq$ 10-ominoes are polyominoes with between seven and nine squares: septominoes, octominoes, and nonominoes. For these cases, Aichholzer et al. [ABD$^+$18, Fig. 17] did an exhaustive enumeration of which *tree-shaped* polyominoes cannot fold into a unit cube in a more restrictive ***grid + diagonals model*** (the two middle rows of Table 1), which is identical to the half-grid model above except that every crease has endpoints whose coordinates are integers.

Therefore the only remaining unsolved tree-shaped cases for the half-grid model are exactly these examples not foldable in the grid + diagonals model. Aichholzer et al. [ABD$^+$18, Fig. 17] lists twelve septominoes, three octominoes, and just one nonomino with the property that some cutting into a tree-shaped polyomino has no grid folding. Figures 1, 2, and 3 list all tree-shaped cuttings of these polyominoes that lack a grid folding, as computed by Aichholzer for [ABD$^+$18] (but which have not previously appeared).

## 2 Results

In this paper, we show how to fold the one nonomino case (the $3 \times 3$ square), one of the octomino cases (the $2 \times 4$ square), and one of the septomino cases (the U)

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[1] A small typo in [ABD$^+$18] is that the Introduction fails to mention the half-grid nature of this result, though their Theorem 3 correctly states the result. Their result also guarantees that every face of the cube is covered by a seamless square in the folding, a property we ignore here.

[2] The Introduction of [ABD$^+$18] claims to "characterize all the polyominoes that can be folded into a unit cube, in grid-based models", but in fact the characterizations for $< 10$ and $\geq 10$ squares are in two different models, as we now detail, so neither is a complete characterization.

| Coordinates | Creases | Polyominoes | Polyomino sizes and results |
|---|---|---|---|
| Grid | Orthogonal | Tree-shaped | Characterized $\leq 14$ [ABD+18]<br>Characterized height-2 and height-3 [ABD+18]<br>OPEN: $\geq 15$ of height $\geq 3$ |
| Grid | Orthogonal | Arbitrary | Partially characterized [AAC+19]<br>OPEN: $\geq 7$ |
| Grid | Diagonal | Tree-shaped | Characterized $\leq 14$; all $10, 11, 12, 13, 14$ [ABD+18]<br>OPEN: $\geq 15$ |
| Grid | Diagonal | Arbitrary | OPEN: $\geq 7$ |
| Half-grid | Diagonal | Tree-shaped | **All $\geq 9$ [this paper]**<br>OPEN: $7, 8$ |
| Half-grid | Diagonal | Arbitrary | All $\geq 10$ [ABD+18]<br>OPEN: $7, 8, 9$ |

**Table 1:** Summary of known/open characterizations of which polyominoes fold into a unit cube in six different models, according to whether crease endpoints must be integers ("grid") or can be half-integers ("half-grid"); whether creases must be horizontal and vertical ("orthogonal") or they can also be at $\pm 45°$ ("diagonal"); and whether the polyominos' duals must be trees ("tree-shaped") or can have cycles ("arbitrary"). Numbers (between 7 and 15) refer to the number of squares in the polyomino, except that "height" refers to the smaller dimension of the polyomino's bounding box. "All" means that all polyominoes of a given size fold into a unit cube; "Characterized" means that there is a list of which do and which do not; "Partially characterized" means that there are necessary conditions and sufficient conditions.

into a cube in the half-grid model. Because our foldings do not require any particular cuts, they also work for any tree-shaped polyomino resulting from cutting these polyominoes (the shaded cases in Figures 1, 2, and 3). In particular, our solution to the sole nonomino case implies (together with [ABD+18, Theorem 3]) that all tree-shaped polyominoes with at least nine squares fold into a cube in the half-grid model.

Figures 4, 5, and 6 show how to fold a $3 \times 3$ square, $2 \times 4$ rectangle, and U, respectively, into a unit cube. In the crease patterns of subfigures (a), dotted lines indicate the integer grid, while solid lines indicate creases and paper boundary. Mountain creases are drawn in red, valley creases are drawn in blue, and crease lines are partially transparent if they fold by $\pm 90°$ and fully opaque if they fold by $\pm 180°$. This notation enables verification of the folding via Origami Simulator [GDG18], which generated the intermediate 3D foldings in subfigures (c). (As Figure 6(c) makes clear, the simulation allows collisions and material stretch during the motion, but it still verifies the final folding.) Subfigures (b) present human-drawn views of the folded states with the faces spread out slightly to make clear how the faces can be stacked while avoiding collision. In addition to the full folded state with translucent faces (right), which reveals mainly the front three faces of the cube, we show the subfolding of just the back three faces of the cube (left). To show the correspondence between the crease pattern (a) and folded state (b), we also label the faces that make up the outer cube surface.

The foldings in Figures 4 and 5 shift the grid of the polyomino by $\frac{1}{2}$ to make the grid of the unit cube. Curiously, the folding of the U septomino in Figure 6 does not, and barely uses the half-grid model by having two crossing diagonals which meet at a half-integer point.

## 3 Other Related Work

Beyond the problem studied in this paper, several other variations have been considered.

In addition to the results mentioned above, Aichholzer et al. [ABD+18] studied the ***grid model*** where creases must be horizontal or vertical (no diagonals) and have endpoints at integer coordinates (the top two rows of Table 1). Specifically, they characterized exactly which tree-shaped polyominoes of height 2 or 3 (i.e., fitting in a $2 \times \infty$ or $3 \times \infty$ strip) fold into a unit cube; their condition can be checked in linear time. A general characterization remains open. They also proved separations between the models in Table 1, along with a few other models, and characterized which polyiamonds (edge-to-edge joinings of equilateral triangles) fold into a unit tetrahedron in an analog to the grid model.

At CCCG 2019, Aichholzer et al. [AAC+19] considered the grid model when the polyomino has holes (and is thus not tree-shaped). This case corresponds to some puzzles invented by Nikolai Beluhov [Bel14] which originally motivated [ABD+18] as well. Aichholzer et al. [AAC+19] gave sufficient conditions when a polyomino containing certain hole shapes can fold into a cube, as well as some necessary conditions, but a general characterization remains open.

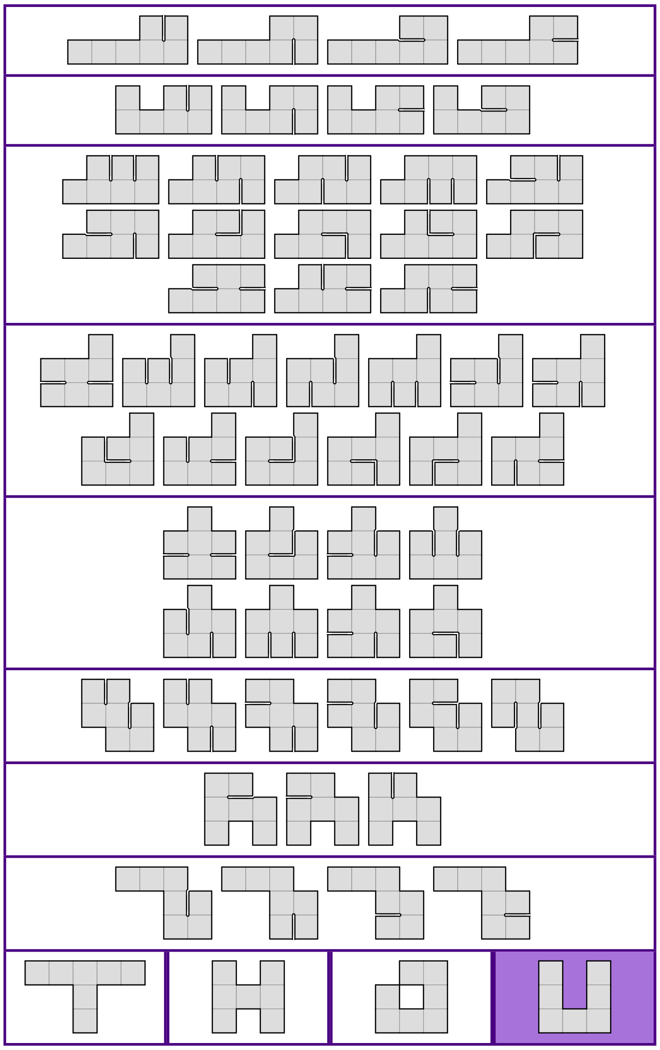


**Figure 1:** Tree-shaped septominoes that cannot fold into a unit cube in the grid + diagonals model, as computed by Aichholzer [ABD+18]. Figure 6 shows how to fold the shaded case in the half-grid model.

Gardner [Gar95] posed a puzzle about cutting the $3 \times 3$ square with slits and then folding along orthogonal grid lines into a unit cube with the additional property of just one side of the paper showing on the outside. Gardner gave one solution, and stated that it can be done "in many different ways". Dunham and Whieldon [DW17] subsequently found all solutions (with and without the additional property) by exhaustive search.

Off the polyomino grid, Catalano-Johnson, Loeb, and Beebee [CLB01] (see also [DO07, Section 15.4.1]) proved that the smallest square that folds into a unit cube has dimensions $(2\sqrt{2}) \times (2\sqrt{2}) \approx 2.8284 \times 2.8284$. This folding implies a folding of a $3 \times 3$ square into a unit cube: just fold away the extra material first. Our innovation is to show that there is a folding in the half-grid model. By contrast, the solution in [CLB01] rotates the square 45° to make the grid of the unit cube.

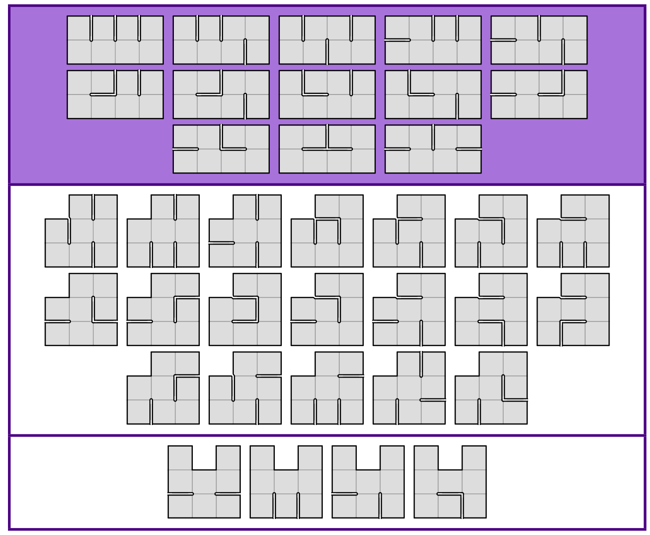


**Figure 2:** Tree-shaped octominoes that cannot fold into a unit cube in the grid + diagonals model, as computed by Aichholzer [ABD+18]. Figure 5 shows how to fold the shaded cases in the half-grid model.

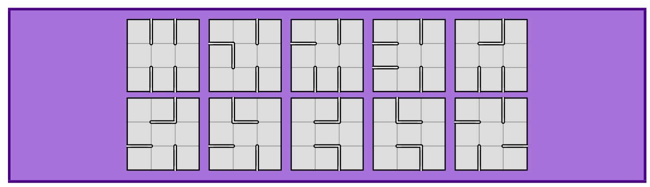


**Figure 3:** Tree-shaped nonominoes that cannot fold into a unit cube in the grid + diagonals model, as computed by Aichholzer [ABD+18]. Figure 4 shows how to fold all of them in the half-grid model.

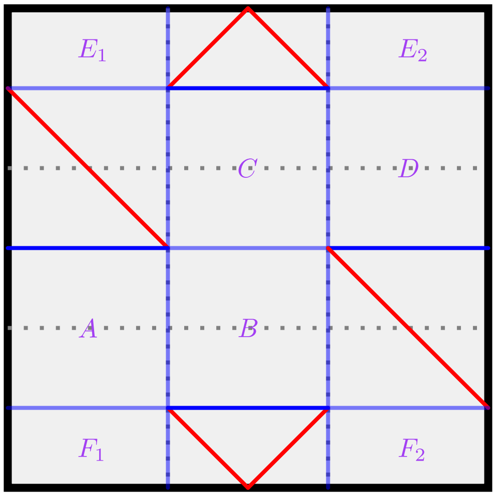
## 4 Open Problems

We conjecture that the remaining septomino and octomino cases cannot be folded into a cube, but could not find an easy argument for impossibility. The best approach may be an exhaustive search for foldings in the half-grid model.

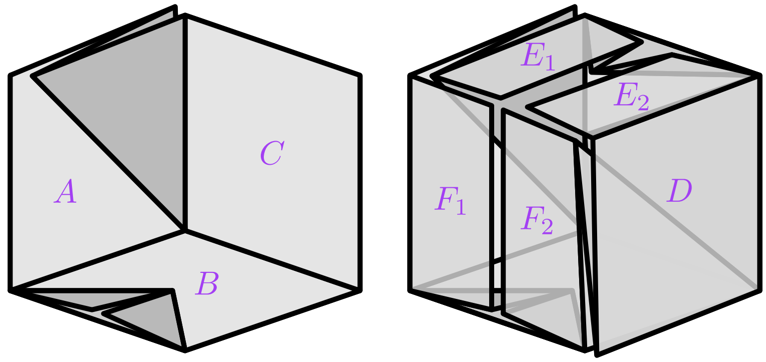
Beyond just tree-shaped polyominoes, we conjecture that all polyominoes with at least nine squares fold into a cube in the half-grid model. The result for at least ten squares [ABD+18, Theorem 3] does not rely on the tree-shaped property, but the existence of grid foldings for all nonominoes beyond the $3 \times 3$ square does. Thus we would need to verify that all 438 non-tree-shaped nonominoes fold into a cube, which we have started to do, but may be easiest to complete via exhaustive search.

There are also countless other models and additional conditions to consider. We mention a few now.

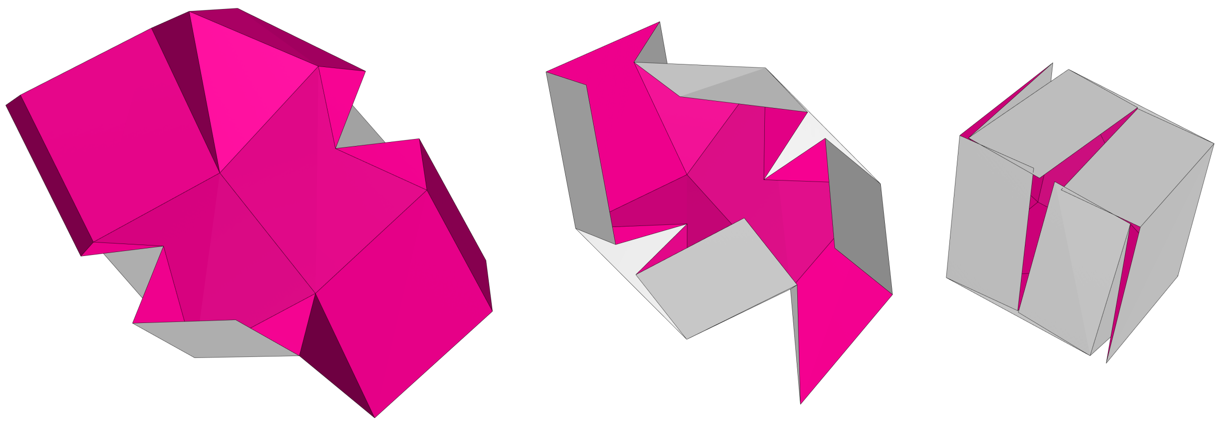
As mentioned in Footnote 1, the universal folding for at least ten squares [ABD+18, Theorem 3] guarantees that every face of the cube is covered by a seamless unit




**(a)** Crease pattern




**(b)** Folded state (right) and back faces (left)



**(c)** Folding animation from Origami Simulator [GDG18]

**Figure 4:** Folding the $3 \times 3$ square into a unit cube.

square. Our folding of the U septomino in Figure 6 shares this property, but we conjecture that this property is unattainable for the $2 \times 4$ rectangle or $3 \times 3$ square because (unlike the U) they seem to need to misalign the cube's grid with the polyomino's grid.

Gardner's $3 \times 3$ puzzle [Gar95] mentioned in Section 3 required that the surface of the cube be made entirely from the same side of the piece of paper. Our foldings of the $3 \times 3$ square (Figure 4) and $2 \times 4$ rectangle (Figure 5), while our folding of the U (Figure 6) does not, and we conjecture that it cannot. With this restriction, many other problems become open again. For example, can all polyominoes of at least ten squares still fold into a unit cube?

Finally, the animations we draw in Figures 4(c), 5(c), and 6(c) raise the question of which cube foldings are achievable as ***rigid origami*** (avoiding collisions while folding only at creases). This direction has yet to be explored.

## Acknowledgments

This work was initiated at the 9th Annual OrigaMIT Convention held at MIT on November 9, 2019, where multiple groups tackled the then-unsolved puzzles posed in [DD19], which are now solved by Figure 4.

We thank Oswin Aichholzer for providing the data from [ABD+18] that enabled us to draw Figures 1, 2, and 3.

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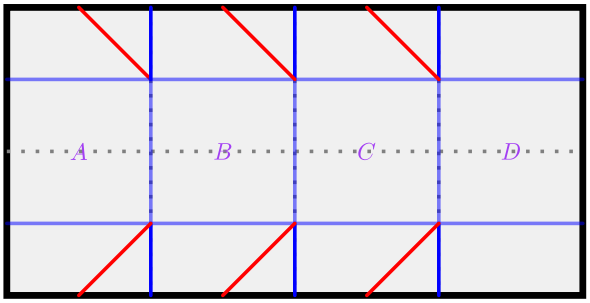


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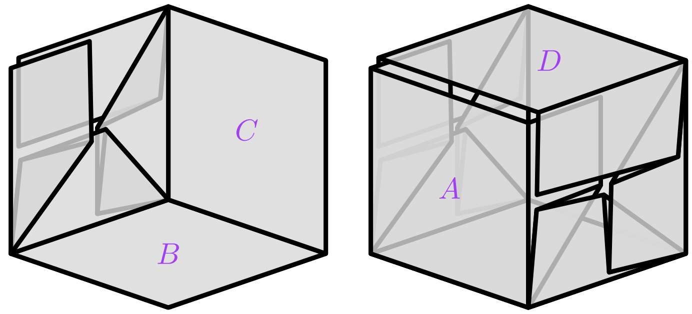
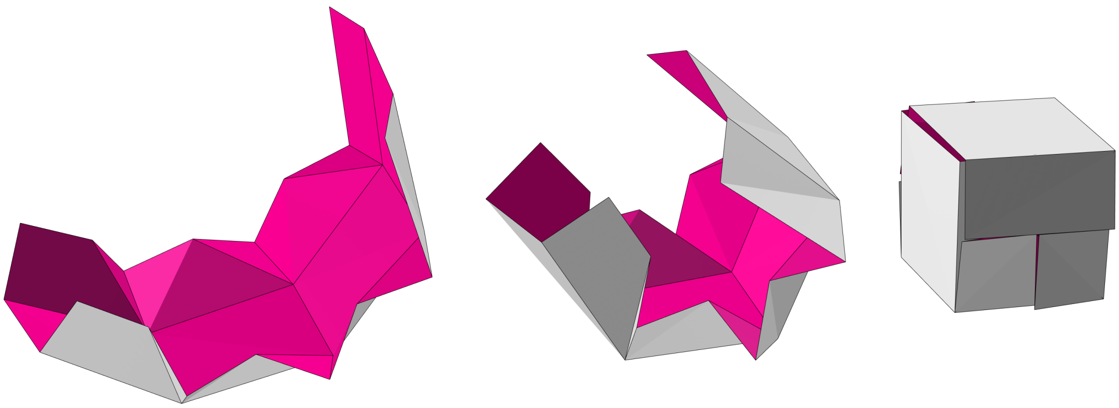



**(b)** Folded state (right) and back faces (left)



**(c)** Folding animation from Origami Simulator [GDG18]

**Figure 5:** Folding the $2 \times 4$ rectangle into a unit cube.

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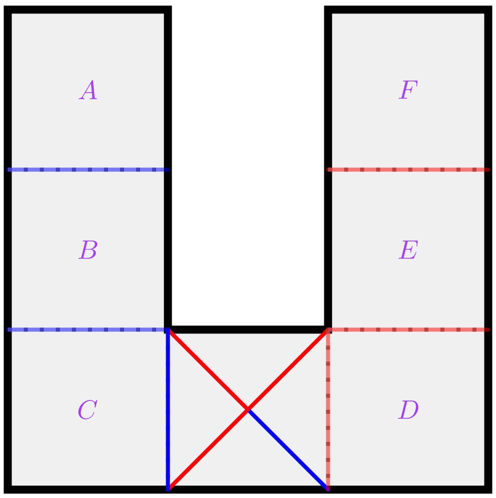


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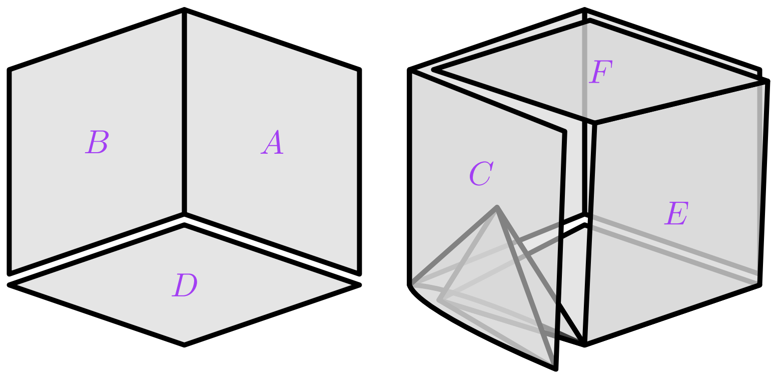


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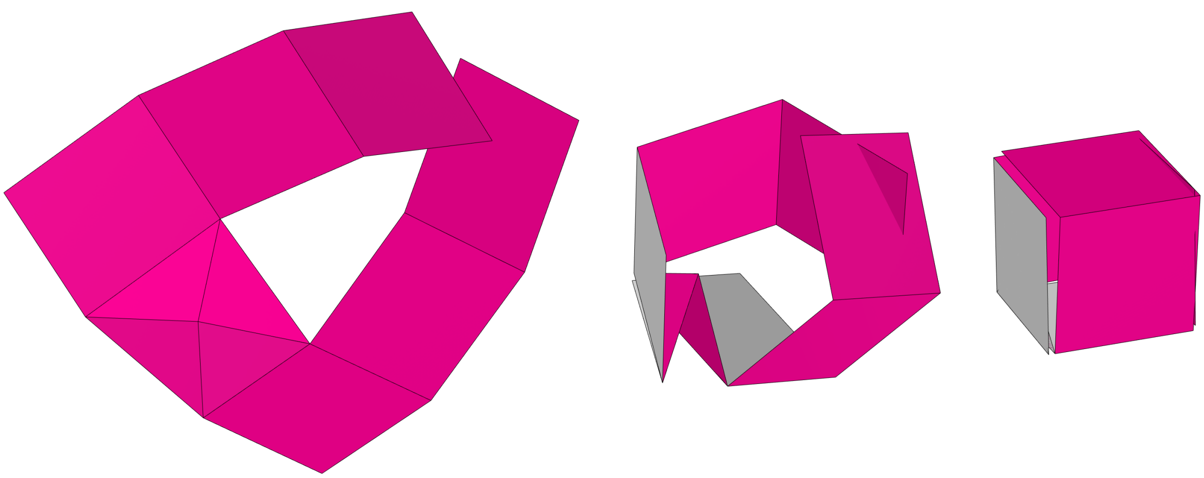

**(c)** Folding animation from Origami Simulator [GDG18]

**Figure 6:** Folding the U septomino into a unit cube.