# Discretization to Prove the Nonexistence of "Small" Common Unfoldings Between Polyhedra

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## Abstract

We show that no $< 300$-gon is a common unfolding between any two doubly covered triangles whose angles are rationally independent algebraic numbers. Here an unfolding of a polyhedron is a polygon obtained by cutting anywhere on the polyhedron's surface and unfolding it.


## 1 Introduction

An ***unfolding*** of a polyhedron $\mathcal{Q}$ is a simple polygon obtained from $\mathcal{Q}$ by cutting anywhere on the surface and unfolding it flat. A ***common unfolding*** between two polyhedra $\mathcal{Q}^0$ and $\mathcal{Q}^1$ is a polygon that is an unfolding of $\mathcal{Q}^0$ and of $\mathcal{Q}^1$. It is open whether any pair of Platonic solids have a common unfolding [4] (though $O(1)$ "refoldings" suffice [3]). For other classes of polyhedra, there are some positive results showing common unfoldings [1, 2, 4, 5, 6]. However, there are no results proving nonexistence of common unfoldings. In other words, it is not known whether there is a pair of polyhedra having no common unfolding.

One difficulty in proving the nonexistence of common unfoldings is that we cannot check by a simple exhaustive search whether two polyhedra have a common unfolding. When we unfold a convex polyhedron $\mathcal{Q}$ to a simple polygon $\mathcal{P}$, the cutting lines on the surface form a tree structure spanning all vertices of $\mathcal{Q}$, called the ***cutting tree***. A cutting tree can have vertices and edges anywhere on the surface of $\mathcal{Q}$. Thus there are uncountably many cutting trees, and the number of obtained unfoldings is also uncountable.

We develop a new algorithmic method to prove the nonexistence of common unfoldings, when we bound the number of vertices in the unfolding, between two polyhedra in the class of doubly covered triangles whose angles are rationally independent algebraic numbers.

In Section 2, we define unfolding and the class of polyhedral which we handle in this paper.

In Section 3.1, we show necessary properties of any common unfolding $\mathcal{P}$ between polyhedra $\mathcal{Q}^0$ and $\mathcal{Q}^1$. First, we consider a correspondence between the boundary of $\mathcal{P}$ on $\mathcal{Q}$ when a polyhedron $\mathcal{Q}$ is unfolded to a polygon $\mathcal{P}$. Next, we define automorphism maps on the boundary, which are called ***gluing maps***, induced by two ways of gluing when $\mathcal{P}$ is folded into $\mathcal{Q}^0$ and $\mathcal{Q}^1$. Finally, we focus on sequences of points on the boundary of the polygon, which are called ***spreading sequences*** and have an essential role in common unfoldings.

In Section 3.2, we introduce a form form of common unfoldings. First, we define the standard-form common unfolding using the notion ad sequence. Next, we show that it is sufficient to consider only standard-form common unfoldings for checking the existence of common unfolding. Finally, we show that the number of standard-form common unfoldings is finite for a given number of vertices in the unfolding. Moreover, we give an algorithm to enumerate the candidates of standard-form common unfoldings.

In Section 3.3, we give a necessary condition and an algorithm to decide whether a candidate standard-form common unfolding represented by a sequence of angles is feasible.

We implement these algorithms and show that, for $n < 300$, there is no $n$-gon that is a common unfolding between any two doubly covered triangles whose angles are algebraic and rationally independent.

## 2 Preliminaries

We consider the common unfolding between two doubly covered triangles (DCT). DCT is a class of polyhedra made by gluing the corresponding edges of two copies of a triangle; see Figure 1. It can be regarded as a kind of polyhedron whose volume is zero. Let $\mathcal{Q}^0$ and $\mathcal{Q}^1$ be

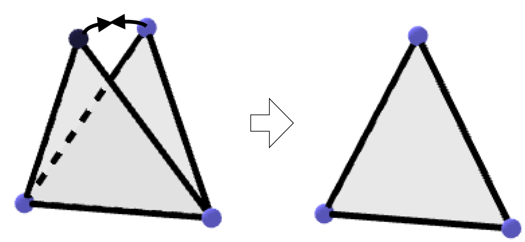

Figure 1: Doubly covered triangle.

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a pair of DCTs and vertices of $\mathcal{Q}^i$ be $v_0^i, v_1^i$, and $v_2^i$. We define the sum of angles gathering at $v_j^i$ by $\theta_j^i$. In other words, the angle on a face of $\mathcal{Q}^i$ is $\frac{\theta_j^i}{2}$; see Figure 2. We assume that edge lengths of $\mathcal{Q}^0$ and $\mathcal{Q}^1$ are adjusted so that the surface areas are the same because it is trivially necessary.

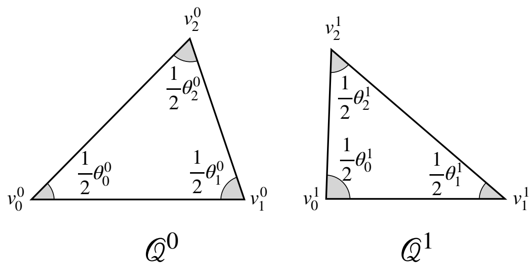


Figure 2: The interior angles of $\mathcal{Q}^0$ and $\mathcal{Q}^1$.

Moreover, we impose the following restrictions on the angles of $\mathcal{Q}^0$ and $\mathcal{Q}^1$.

1. A DCT $Q^i$ is ***algebraic*** if $\theta_0^i$ and $\theta_1^i \in \mathbb{Q}^*$ where $\mathbb{Q}^*$ is the algebraic closure on $\mathbb{Q}$. (Here we note that $\theta_2^i = 2\pi - (\theta_0^i + \theta_1^i)$, and $\theta_2^i \notin \mathbb{Q}^*$ if $\theta_0^i, \theta_1^i \in \mathbb{Q}^*$.)

2. A pair of DCTs $\mathcal{Q}^0$ and $\mathcal{Q}^1$ are ***(rationally) independent*** if $\forall m_i(\neq 0) \in \mathbb{Q}$, $m_0\theta_0^0 + m_1\theta_1^0 + m_3\theta_0^1 + m_4\theta_1^1 \neq 0$.

Hereafter, we assume that $\mathcal{Q}^0$ and $\mathcal{Q}^1$ are algebraic and independent. Therefore, each of $\mathcal{Q}^0$ and $\mathcal{Q}^1$ is not an isosceles triangle and has no angle that is a rational multiple of $\pi$. Here we note that we introduce these restrictions not to avoid a counterexample but to support the proof technique. We treat $\theta_j^i$ as symbols and do not care about the concrete values until Section 3.3. When we consider an assignment of the values of $\theta_j^i$, we use map $\lambda : \{\theta_0^0, \theta_1^0, \theta_2^0, \theta_0^1, \theta_1^1, \theta_2^1\} \to \mathbb{R}_{>0}$.

**Example 1** *If* $(\lambda(\theta_0^0), \lambda(\theta_1^0), \lambda(\theta_2^0), \lambda(\theta_0^1), \lambda(\theta_1^1), \lambda(\theta_2^1)) = (\sqrt{2}, \sqrt{3}, 2\pi - \sqrt{2} - \sqrt{3}, \sqrt{5}, \sqrt{7}, 2\pi - \sqrt{5} - \sqrt{7})$, $\mathcal{Q}^0$ *and* $\mathcal{Q}^1$ *are algebraic and independent.*

When we unfold a polyhedron $\mathcal{Q}$ to a polygon $\mathcal{P}$, cutting lines on the surface form a tree structure [4]. We denote it by $\mathcal{T}$. Conversely, points on the boundary of $\mathcal{P}$ are glued and make a point on $\mathcal{T}$ when we fold $\mathcal{P}$ to $\mathcal{Q}$. We call it a ***folding map*** and write it by $f : \partial\mathcal{P} \to \mathcal{T}$ where $\partial\mathcal{P}$ is the boundary of $\mathcal{P}$; see Figure 3.

Let $\mathcal{P}$ be the unfolding of a DCT $\mathcal{Q}$ by $\mathcal{T}$. The topology of $\mathcal{T}$ can be classified into two cases, as illustrated in Figure 4: a ***Y-form*** is a tree with a single point $b^i$ of degree 3 (and with leaves at the vertices of $\mathcal{Q}$), and a ***V-form*** is just a path (through all vertices of $\mathcal{Q}$).

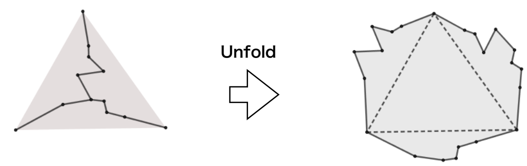


Figure 3: Unfolding a polyhedron.

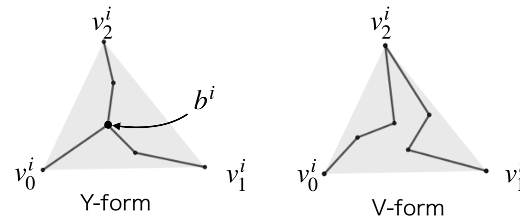


Figure 4: Topologies of cutting trees of doubly covered triangles.

## 3 Nonexistence of Small Common Unfoldings for $\mathcal{Q}^0$ and $\mathcal{Q}^1$

In this section, we assume there is a polygon $\mathcal{P}$ that is a common unfolding of $\mathcal{Q}^0$ and $\mathcal{Q}^1$ by $\mathcal{T}^0$ and $\mathcal{T}^1$ with folding maps $f^0$ and $f^1$.

Hereafter, we consider only the case that both $\mathcal{T}^0$ and $\mathcal{T}^1$ are Y-form. It can be shown that in other cases existence of a common unfolding would contradict our assumption that $\mathcal{Q}^0$ and $\mathcal{Q}^1$ are algebraic and independent (see the proof in Appendix A).

### 3.1 Gluing Map

On $\partial\mathcal{P}$, there are three points $l_0^i, l_1^i, l_2^i$ corresponding to $v_0^i, v_1^i, v_2^i$, such as $f^i(l_j^i) = v_j^i$. Moreover, there are three points $m_0^i, m_1^i, m_2^i$ corresponded to $b^i$, such as $b^i = f^i(m_0^i) = f^i(m_1^i) = f^i(m_2^i)$. We define $L^i := \{l_0^i, l_1^i, l_2^i\}$ and $M^i := \{m_0^i, m_1^i, m_2^i\}$; see Figure 5. Let $I_j^i$ be the intervals on $\partial\mathcal{P}$ between $m_j^i, m_{j+1}^i$. The following holds.

**Observation 2** *For* $p \in \partial\mathcal{P}$, *let* $\alpha(p)$ *be the interior angle at* $p$.

- $\alpha(l_j^i) = \theta_j^i$.
- $\alpha(m_0^i) + \alpha(m_1^i) + \alpha(m_2^i) = 2\pi$.

Without loss of generality, we can assume that $l_j^i$ and $m_j^i$ appear in counterclockwise order $m_0^i, l_0^i, m_1^i, l_1^i, m_2^i, l_2^i$ around $\partial\mathcal{P}$ for each $i = 0, 1$.

**Definition 3** *We define a gluing map* $gl^i : \partial\mathcal{P} \to \partial\mathcal{P}$ *by the map returns the point to which is glued by the mapping as follows.*

- *If* $p \in L^i \cup M^i$, *then* $gl^i(p) := p$.

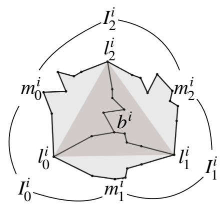


Figure 5: $L^i = \{l_0^i, l_1^i, l_2^i\}$, $M^i = \{m_0^i, m_1^i, m_2^i\}$.

- *Otherwise,* $gl^i(p) := p'$ *such that* $f^i(p) = f^i(p')$; $p'$ *is determined uniquely.*

**Observation 4** *Let* $p \notin L^i \cup M^i$. *The following holds:*

- $p \in I_j^i \Rightarrow gl^i(p) \in I_j^i$.
- $p \in I_j^i \Rightarrow l_j^i$ *is the midpoint of* $(p, gl^i(p))$ *on* $\partial\mathcal{P}$.
- $\alpha(p) + \alpha(gl^i(p)) = 2\pi$ *where* $\alpha(p)$ *is the interior angle of* $p \in \partial\mathcal{P}$.

**Definition 5** *(spreading sequence* $spr(l_j^i)$*)*
*For each* $l_j^i \in L^i$, *we define the spreading sequence* $spr(l_j^i)$ *by the sequence of points obtained by alternative iterations of* $gl^i$ *and* $gl^{i+1}$,

$$(l_j^i, gl^{i+1}(l_j^i), gl^i(gl^{i+1}(gl_j^i)), gl^{i+1}(gl^i(gl^{i+1}(l_j^i)), \dots),$$

*until same points are repeated.*[1] *In other words, a spreading sequence ends at a point in* $\bigcup_{i,j} L^i \cup M^i$.

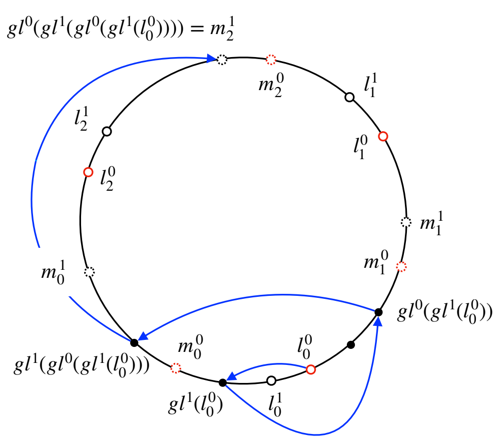


Figure 6: The spreading sequence of $l_0^0$.

**Observation 6** *The interior angles of odd-numbered points of* $spr(l_j^i)$ *are* $\theta_j^i$, *and even-numbered ones are* $\overline{\theta_j^i}$ *where* $\overline{\theta_j^i} := 2\pi - \theta_j^i$.

[1] The superscript indices are taken modulo 2 in this paper. Specifically, $gl^{i+1}$ means $gl^0$ for $i = 1$ because $gl^i$ defined for $i = 0, 1$.

**Lemma 7** *If* $i \neq i'$ *or* $j \neq j'$, $spr(l_j^i)$ *and* $spr(l_{j'}^{i'})$ *share no point.*

**Proof.** If a point appears in both of $spr(l_j^i), spr(l_{j'}^{i'})$, by Observation 7, $\theta_j^i = \theta_{j'}^{i'}$, $\theta_j^i = \overline{\theta_{j'}^{i'}}$, $\overline{\theta_j^i} = \theta_{j'}^{i'}$, or $\overline{\theta_j^i} = \overline{\theta_{j'}^{i'}}$ holds. In any case, it contradicts the independence of the angles. □

**Lemma 8** *For any* $l_j^i \in L^i$, *the length of* $spr(l_j^i)$ *is finite.*

**Proof.** Because the angles are algebraic and independent, $\theta_j^i \neq \pi$. It means that all points included in some spreading sequence are vertices of $\mathcal{P}$. By the definition of the spreading sequence, a point does not appear twice or more in a spreading sequence. Therefore if there is a spreading sequence whose length is infinite, it produces infinite vertices of $\mathcal{P}$. It is a contradiction. □

**Lemma 9** *For any* $l_j^i \in L^i$, *there exists unique* $m_k^{i+1} \in M^{i+1}$ *such that* $spr(l_j^i) = (l_j^i, \dots, m_k^{i+1})$.

**Proof.** The endpoint of a spreading sequence belongs to $M^0 \cup M^1 \cup L^0 \cup L^1$. If the endpoint belongs to $L^0$ or $L^1$, it contradicts the independence of the angles. Therefore, the endpoints belong to $M^0 \cup M^1$. Inversely, each of $M^0 \cup M^1$ is the endpoint of some spreading sequence because the numbers of $L^0 \cup L^1$ and $M^0 \cup M^1$ are the same. Let us consider the spreading sequences that end at $m_0^0, m_1^0$, or $m_2^0$. The sum of the angles of $m_0^0, m_1^0$, or $m_2^0$ must be $2\pi$, and it will be realized by only $\theta_0^0 + \theta_1^0 + \theta_2^0$ and $\theta_0^1 + \theta_1^1 + \theta_2^1$ by their independence. (Note that $\overline{\theta_0^i} + \overline{\theta_1^i} + \overline{\theta_2^i} = 6\pi - (\theta_0^i + \theta_1^i + \theta_2^i) = 4\pi \neq 2\pi$.) Therefore, the length of each of the spreading sequences is odd by Observation 6. By considering the parity, we can see that these spreading sequences must start from $l_0^1, l_1^1$, or $l_2^1$. □

**Lemma 10** *Let* $S_j^i := \{p : p \in spr(l_j^i)\}$.
*Then* $\bigcup_{i,j} S_j^i$ *divides into* $\partial\mathcal{P}$ *equilateral intervals.*

**Proof.** Let $d_+(p)$ and $d_-(p)$ be the distance between $p$ and its counterclockwise and clockwise nearest point of $\bigcup_{i,j} S_j^i$ respectively. We prove that $d_+(p)$ and $d_-(p)$ are uniform for any $p$ in $\bigcup_{i,j} S_j^i$. Let $s \in \bigcup_{i,j} S_j^i$ be the clockwise nearest point of $m_0^0$, and $c := d_-(m_0^0)$; see Figure 7. Let take $l_j^1 \in L^1$ such that $spr(l_j^1) = (l_j^1, \dots, m_0^0)$. If there is a point $p' \in S_j^1$ such that $d_+(p') = c' < c$ or $d_-(p') = c' < c$, by using Observation 4 inductively, there is a point $p''$ such that the distance between $p'', m_1^0$ is $c'$; see Figure 8. It contradicts that $s$ is the nearest. Therefore, $c = d_+(p) = d_-(p)$ for any point $p \in S_j^1$. Especially, $d_+(m_0^0) = c$. Next, we focus on $d_+(m_1^0), d_-(m_1^0), d_+(m_2^1)$, and $d_-(m_2^1)$. It is easy to see that $d_+(m_0^0) = d_-(m_1^0), d_+(m_1^0) = d_-(m_2^0), d_+(m_2^0) = d_-(m_0^0)$; see Figure 7. Thus, we can check that $c =$

$d_+(p) = d_-(p)$ for any point $p \in \bigcup_j S_j^0$ by repeating the same discussion for $m_1^0$ and $m_2^0$. There exist $p \in \bigcup_j S_j^0, p' \in \bigcup_j S_j^1$ such that $p$ and $p'$ are adjacent, and they share the distance to the nearest. Therefore, $c = d_+(p) = d_-(p)$ for any point $p \in \bigcup_j S_j^1$. □

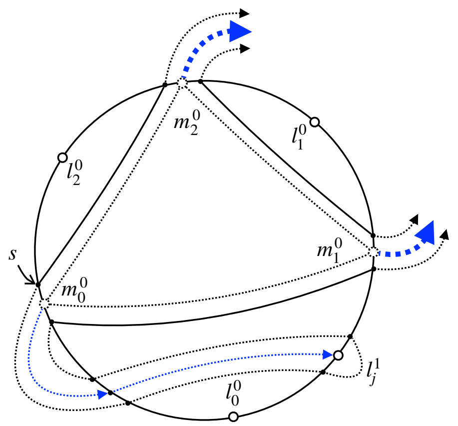


Figure 7: The nearest distances next to $m_j^i$.

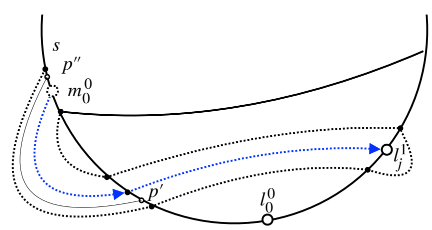


Figure 8: $c = d_+(p) = d_-(p)$ for any point $p \in S_j^1$.

## 3.2 standard

**Definition 11** *If all vertices of $\mathcal{P}$ are included in $\bigcup_{i,j} S_j^i$, we call $\mathcal{P}$ is a standard-form common unfolding.*

**Lemma 12** *If $\mathcal{Q}^0$ and $\mathcal{Q}^1$ have a common unfolding, $\mathcal{Q}^0$ and $\mathcal{Q}^1$ have a standard-form common unfolding.*

**Proof.** By Lemma 10, the points of $\bigcup_j S_j^0$ and $\bigcup_j S_j^1$ are lined up alternately on $\partial\mathcal{P}$. Let take a pair of adjacent points and $m$ be the interval between them. Let $(p_0, p_1, p_2, \ldots, p_k)$ be the vertices of $\mathcal{P}$ on $m'$. Because $m$ is glued to another interval $m'$, $(p_0, p_1, p_2, \ldots, p_k)$ make vertices $(p'_0, p'_1, p'_2, \ldots, p'_k)$ such that $\alpha(p_i) = 2\pi - \alpha(p'_i)$. In the same way as the proof of Lemma 10, it spreads into all intervals. On the boundary of $\mathcal{P}$ except $\bigcup_{i,j} S_j^i$, the interior angles are $\alpha(p_0), \ldots, \alpha(p_k)$ and $2\pi - \alpha(p_k), \ldots, 2\pi - \alpha(p_0)$ alternately; see Figure 9. We focus on the cutting tree $\mathcal{T}$ into one side polyhedron. Let $\mathcal{T}'$ be the cutting tree replacing each interval of $\mathcal{T}$ with a straight line segment. $\mathcal{T}'$ is kept the interior angles at $\bigcup_{i,j} S_j^i$; see Figure 10. Let $\mathcal{P}'$ be the unfolding by $\mathcal{T}'$. Then $\mathcal{P}'$ is a standard-form common unfolding of $\mathcal{Q}^0$ and $\mathcal{Q}^1$. □

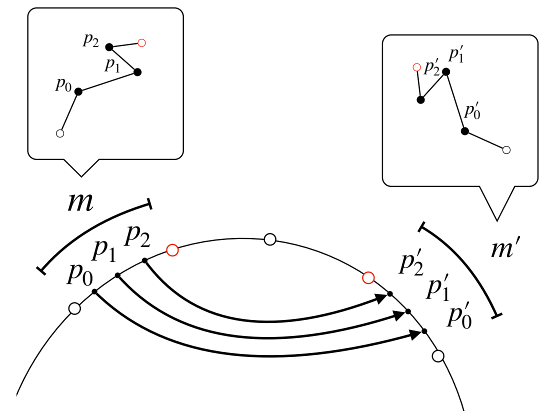


Figure 9: $(p_0, p_1, \ldots, p_m)$ on the interval $m$.

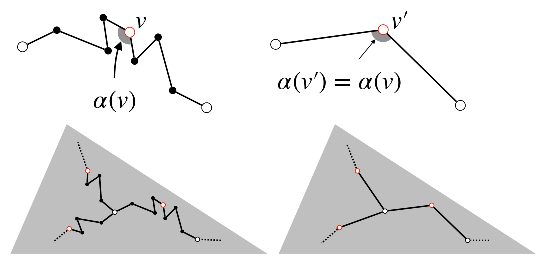


Figure 10: The reduction of a common unfolding into a standard-form common unfolding.

By Lemma 12, if there is no standard-form common unfolding between two polyhedra, there is no common unfolding.Therefore, we can search the common unfolding in the standard-form common unfoldings, whose edges are isometric and vertices are included in $\bigcup_{i,j} S_j^i$. The standard-form common unfoldings are represented by a sequence of interior angles. By fixing $n$, we can enumerate the sequences of interior angles of length $n$ to be candidates of standard-form common unfolding. Details of the algorithm are given in Algorithm 1. Because the length of each spreading sequence is odd, $n$ should be an integer that is not a multiple of 4 but even. First, we prepare a cyclic array of length $n$ to store the interior angles. Next, we choose six array positions to store the interior angles of $l_j^i$. It causes $O(n^5)$ combinations. Next, we compute the spreading sequences and determine the interior angles. If distinct angles are assigned to one point, we return to the step of choosing positions of $l_j^i$. After the placement of $l_j^i$ is determined, the construction of the spreading sequences takes $O(n)$

time because the length of each spreading sequence is at most $n$. If we obtain a feasible array, we output this one as the candidate of a standard.

**Algorithm 1:** Enumerating candidate angle squares for standard-form common unfoldings

**input :** The number of vertices $n$
**output:** Sequences of interior angles
1 Let $C$ be a cyclic array of length $n$.
2 $m_0^0 := 0$
3 **forall** $m_1^0, m_2^0, m_0^1, m_1^1, m_2^1$ such that $0 = m_0^0 < m_1^0 < m_2^0 < n, 0 < m_0^1 < m_1^1 < m_2^1 < n$ **do**
4   **for** $i = 0, 1$ and $j = 0, 1, 2$ **do**
5     **if** $m_{j+1}^i - m_j^i$ are odd **then**
6       Return to line 3.
7     **end**
8     $l_j^i := m_j^i + \frac{1}{2}(m_{j+1}^i - m_j^i) \bmod n$
9   **end**
10   Define $gl^0, gl^1$ by Definition 3.
11   **for** i = 0,1 and j = 0, 1, 2 **do**
12     $p := l_j^i$
13     $k := (j + 1) \bmod 2$
14     $C[l_j^i] := \theta_j^i$.
15     **while** $p \neq gl^k(p)$ **do**
16       $p := gl^k(p)$
17       **if** $C[p]$ is not yet defined **then**
18         **if** $k = 1$ **then**
19           $C[p] := \theta_j^i$
20         **else**
21           $C[p] := \overline{\theta_j^i}$
22         **end**
23       **else**
24         Return to line 3.
25       **end**
26       $k := (k + 1) \bmod 2$
27     **end**
28   **end**
29   **if** $\{C[m_0^i], C[m_1^i], C[m_2^i]\} = \{\theta_0^{i+1}, \theta_1^{i+1}, \theta_2^{i+1}\}$ for each $i$ **then**
    **output:** C
30   **end**
31 **end**

### 3.3 Checking Polygon Closure

For example, Algorithm 1 outputs the following sequence (see Figure 11):

$$\phi = (\theta_2^1, \theta_2^0, \overline{\theta_2^1}, \theta_2^0, \theta_1^1, \theta_1^0, \theta_0^1, \theta_0^0, \theta_2^1, \overline{\theta_2^0}).$$

It remains to check whether the sequence of interior angles corresponds to a simple polygon. First, we fix the values of $\theta_j^i$ by $\lambda$ like Example 1. We view the polygonal line as lying in the complex plane $\mathbb{C}$. We define an equilateral polygonal line $\mathrm{Poly}_{\phi,\lambda} = (p_0, p_1, \ldots, p_n)$ by the following:

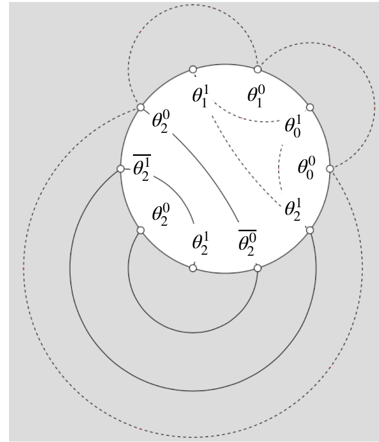

Figure 11: $\phi = (\theta_2^1, \theta_2^0, \overline{\theta_2^1}, \theta_2^0, \theta_1^1, \theta_1^0, \theta_0^1, \theta_0^0, \theta_2^1, \overline{\theta_2^0})$; solid lines represent spreading sequences, and dotted lines connect $m_j^i$

$$p_0 = 1, p_1 = 0 \in \mathbb{C},$$
$$p_{i+1} - p_i = (p_{i-1} - p_i)e^{\sqrt{-1}\,\phi_i}.$$

Here, we remark that $e^{\sqrt{-1}\,\theta} = \cos\theta + \sqrt{-1}\sin\theta$ holds by Euler's Formula. In order to be the common unfolding, $\mathrm{Poly}_{\phi,\lambda}$ must satisfy closure $p_0 = p_n$ and not have self-intersection. We consider only the closure condition of $p_0 = p_n$ because it suffices here to prove the nonexistence of common unfoldings. We can check whether the polygon is closed using the following lemma:

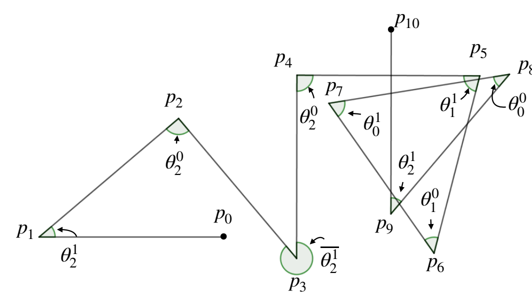

Figure 12: $\mathrm{Poly}_{\phi,\lambda}$ where $\phi = (\theta_2^1, \theta_2^0, \overline{\theta_2^1}, \theta_2^0, \theta_1^1, \theta_1^0, \theta_0^1, \theta_0^0, \theta_2^1, \overline{\theta_2^0})$ and $\lambda\{\theta_j^i\} = (\sqrt{2}, \sqrt{3}, 2\pi - \sqrt{2} - \sqrt{3}, \sqrt{5}, \sqrt{7}, 2\pi - \sqrt{5} - \sqrt{7})$.

**Lemma 13** *For a sequence $\phi = (\phi_0, \phi_1, \ldots, \phi_{n-1})$ of the angles $\theta_j^i$ or $\overline{\theta_j^i}$ and an angle assignment $\lambda$, $\mathrm{Poly}_{\phi,\lambda}$ satisfies $p_0 = p_n$ if and only if the following condition holds:*

(∗) *For each $0 \leq i \leq n$, there exists $j$ uniquely such that $\phi_i + \phi_{i+1} + \cdots + \phi_{j-1} + \phi_j$ is an integer multiple of $2\pi$ and $j - i$ is odd.*

**Proof.** Let $\vec{w}_i$ be the vector along the edge $(p_i, p_{i+1})$. Here, $p_0 = p_n$ is equivalent to $\sum_i \vec{w}_i = 0$. The slope of $\vec{w}_i$ is $\phi_0+\phi_1+\cdots+\phi_i$ or $\phi_0+\phi_1+\cdots+\phi_i-\pi$ depending on whether $i$ is odd or even. Thus, the difference between the slopes of two vectors $\vec{w}_i$ and $\vec{w}_j$ is $\phi_i+\phi_{i+1}+\cdots+\phi_j+\pi$ or $\phi_i + \phi_{i+1} + \cdots + \phi_j$ depending on whether $i$ is odd or even. By the independence of the angles, $\vec{w}_i = -\vec{w}_j$ holds if and only if $j-i$ is odd and $\phi_i+\phi_{i+1}+\cdots+\phi_{j-1}+\phi_j$ is an integer multiple of $2\pi$. It is easy to see that $p_0 = p_n$ if the condition (∗) holds because all vectors are canceled with these inverses. We show $p_0 = p_n$ only if the condition (∗) holds. Let $\vec{w}'_0, \vec{w}'_1, \ldots, \vec{w}'_k$ be the subset of $\vec{w}_0, \vec{w}_1, \ldots, \vec{w}_{n-1}$ choosing without the same or inverse ones. It is sufficient to show $\vec{w}'_0, \vec{w}'_1, \ldots, \vec{w}'_k$ are linearly independent on $\mathbb{Z}$.

We use a classical result on transcendental numbers:

**Theorem 14 (Lindemann's Theorem)** *For any distinct algebraic numbers $a_0, a_1, \ldots, a_m$, the numbers $e^{a_0}, e^{a_1}, \ldots, e^{a_m}$ are linearly independent on $\mathbb{Q}^*$, where $\mathbb{Q}^*$ is the algebraic closure on $\mathbb{Q}$.*

Let $\psi_i$ be the slope of $\vec{w}'_i$; $\vec{w}'_i$ is represented by $e^{\sqrt{-1}\,\psi_i}$. Because we choose $\vec{w}'_0, \vec{w}'_1, \ldots, \vec{w}'_k$ without the same or inverse ones, $\psi_0, \ldots, \psi_k$ are distinct algebraic numbers. Similarly, $\sqrt{-1}\,\psi_0, \ldots, \sqrt{-1}\,\psi_k$ are distinct algebraic numbers. By Lindemann's Theorem, $e^{\sqrt{-1}\,\psi_0}, \ldots, e^{\sqrt{-1}\,\psi_k}$ are linearly independent on $\mathbb{Q}^*$. On $\mathbb{Z}$, they are also linearly independent. Therefore, $e^{\sqrt{-1}\,\psi_0} + e^{\sqrt{-1}\,\psi_1} + \cdots + e^{\sqrt{-1}\,\psi_n} = 0$ only when the condition (∗) holds. □

**Lemma 15** *Whether the condition (∗) holds does not depend on $\lambda$.*

**Proof.** From the independence, the sum of angles is an integer multiple of $\pi$ only if $(\theta_0^0+\theta_1^0+\theta_2^0), (\theta_0^1+\theta_1^1+\theta_2^1)$, or $(\theta_j^i+\overline{\theta_j^i})$. Therefore, whether $\phi_i+\phi_{i+1}+\cdots+\phi_j$ is an integer multiple of $2\pi$ or not depends on only whether they can be divided into the above pairs or not. □

For a given $\phi$, we check that there exists $j$ such that the condition (∗) is satisfied for each $i$ one by one. It can be done in $O(n^2)$ time.

## 4 Computational Experiment

By combining Algorithm 1 and the Lemma 15 technique, we can check that, for given $n$, there is no $n$-gon that is a common unfolding between any two doubly covered triangles whose angles are algebraic and rationally independent. It requires $O(n^7)$ time theoretically. We implemented them and checked that in a range $n < 300$. It takes 1.5 hours in a normal laptop environment (CPU: 1.4GHz Intel Quad-Core i5, OS: macOS 12.4, Memory: 16GB, compiler: GCC $11.3.0_2$, optimize: -O3).

## 5 Conclusion

In this paper, we proved the nonexistence of common unfoldings limited in the number of vertices between two elements in a restricted polyhedral class. The main next step is to remove the limitation on the number of vertices. As you can see from the computational experiments, Lemma 13 requires a strong condition to have a common unfolding. This condition seems not to be satisfied by any sequence obtained by Algorithm 1. If we can prove this conjecture, then we will obtain nonexistence without the limitation on the number of vertices. The extension to polyhedra with more than three vertices would also be interesting. In these cases, there are more possible cutting trees to consider, and we would have to consider how to relate restrictions of the interior angles through the spreading sequences.

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## A Appendix

Here we consider the case that either or both cutting trees are V-form. We assume that at least $\mathcal{T}^0$ is V-form, and that $\mathcal{T}^0$ cuts $v_0^0, v_1^0$ by leaves and spans $v_3^0$ without loss of generality. There are two points $d_0^0, d_1^0$ in the boundary of $\mathcal{P}$ such that $f^0(d_0^0) = f^0(d_1^0) = v_3^0$. Let $L^0 := \{l_0^0, l_1^0\}, M^0 := \emptyset, D^0 := \{d_0^0, d_1^0\}$. If $\mathcal{T}^1$ is also V-form, we define $L^1, M^1, D^1$ in the same manner. Otherwise, we let $L^1 := \{l_0^1, l_1^1, l_2^1\}, M^1 := \{m_0^1, m_1^1, m_2^1\}, D^1 := \emptyset$. We modify the definition of the gluing map.

**Definition 16** *We define $gl^i : \partial\mathcal{P} \to \partial\mathcal{P}$ as follows.*

- *If $p \in L^i \cup M^i \cup D^i$, $gl^i(p) := p$*
- *Otherwise, $gl^i(p) := p'$ such that $f^i(p) = f^i(p')$; $p'$ is determined uniquely.*

We consider the spreading sequences of each $L^0 \cup L^1$. The endpoints belong to $M^i \cup D^i$ by the definition. In both cases, $|L^0 \cup L^1| = |M^0 \cup M^1 \cup D^0 \cup D^1|$. Thus, each of $M^0 \cup M^1 \cup D^0 \cup D^1$ is the endpoint of some spreading sequence. Therefore, $v_3^0$ is made by gluing two points that are the endpoints of some spreading sequences. It means that $\theta_3^0$ is represented by $\theta_j^i + \theta_{j'}^{i'}$, $\theta_j^i + \overline{\theta_{j'}^{i'}}$, or $\overline{\theta_j^i} + \overline{\theta_{j'}^{i'}}$. It contradicts the independence of the angles. Therefore, it is sufficient to consider only the case that both are Y-form.