Open Problems from CCCG 2010

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The following is a description of the problems presented on August 9, 2010 at the open-problem session of the *22nd Canadian Conference on Computational Geometry* held in Winnipeg, Manitoba, Canada.

Coiling Rope in a Box
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Is there a procedure to decide whether a rope of length L and radius r can be coiled to fit in an $a \times b \times c$ box? All five parameters can be assumed to be rational numbers for the decision question. The rope is a smooth curve with a tubular neighborhood of radius $r > 0$, such that the rope does not self-penetrate. In particular, the curve should not turn so sharply that the disks of radius r orthogonal to the curve that determine the tubular neighborhood interpenetrate. For an open curve, each endpoint is surrounded by a ball of radius r.

For a box of dimensions $1 \times 1 \times \frac{1}{2}$ and rope of radius $r = \frac{1}{4}$, perhaps the maximum length achievable is $L = \frac{1}{2} + \frac{\pi}{4} \approx 1.3$, realized by a 'U'-shape as in Figure 1.

Packing circles in a square is a notoriously difficult problem, but perhaps it is easier to pack a rope in a cube, because the continuity of the curve constrains the options.

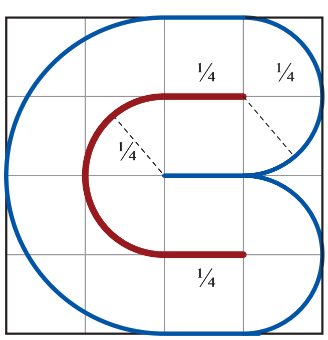

Figure 1: Overhead view of a rope in a box.

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Update. This problem also appeared on MathOverflow,[1] where Greg and Włodzimierz Kuperberg opine that it is open. At the suggestion of several people during the CCCG presentation, the poser started exploring the 2D version. If $k = \frac{1}{2r}$ is an even integer, then there are two natural strategies for coiling the rope within a box whose height renders it two-dimensional, as illustrated in Figure 2. Interestingly, the length of the core rope curve is identical for the two coilings:

$$L = 2(k-1)(r\pi/2) + 2(k-1)^2 r \ .$$

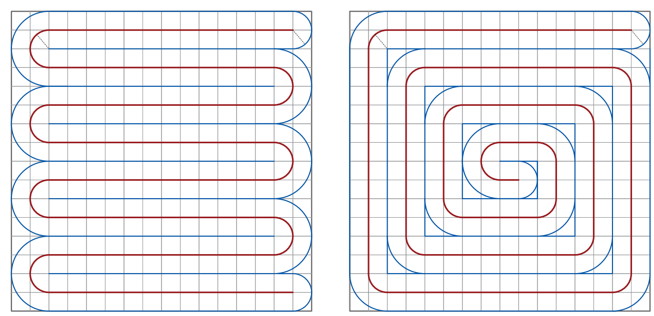

Figure 2: Two 2D coilings in a $1 \times 1 \times 2r$ box. Here $r = \frac{1}{16}$, $k = 8$, and $L = \frac{7\pi}{16} + \frac{49}{8} \approx 7.5$.

When Sticks Fall, Will They Weave?
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Imagine n z-vertical sticks uniformly spaced around a unit-radius circle in the xy-plane. At random times $t_1, t_2, \ldots, t_n \geq 0$, each stick is randomly ε-perturbed from the vertical, and they fall under the influence of gravity. Will some sticks form a "teepee" suspended above the xy-plane?

Let us assume that the sticks are one-dimensional segments of height h, perhaps $h = 2$ so that they span the diameter, and that their base points are pinned to the plane via universal joints. It seems possible that a subset of sticks could fall to form a *weaving* with a cyclic on-top-of graph, as illustrated in Figure 3. Assuming a sufficiently large coefficient of friction μ between pairs of sticks, it

[1] http://mathoverflow.net/questions/26525/.

seems conceivable that such a structure would not collapse to the plane. Is it possible that some sticks form a woven "teepee" structure above the plane? Or would all sticks ultimately flatten to the plane?

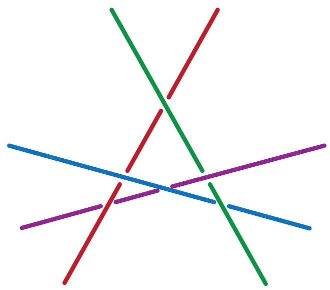

Figure 3: A weaving of four sticks.

Update. This problem also appeared on MathOverflow,[2] where Scott Morrison observed that if all sticks are released at the same time $t = 0$, then they would hit one another with probability zero.

Linkless embeddings of graphs in $\mathbb{R}^3$
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In one of the early papers on linkless embedding, Sachs [Sac83] asked a question that still remains open: is there an analogue of Fáry's theorem for three-dimensional drawing? That is, if a graph has a linkless or flat embedding with curved or polygonal edges, does it automatically have a linkless or flat embedding with straight line segment edges? If so, how can we find these straight drawings efficiently? If not, which graphs do have linkless straight drawings? An embedding of a graph into $\mathbb{R}^3$ is *linkless* if, for every pair of disjoint cycles C_1 and C_2, there is a topological sphere separating C_1 from C_2; and an embedding is *flat* if every cycle in the graph forms the boundary of a topological disk that is disjoint from all the other vertices and edges of the graph. A flat embedding is always linkless, while a linkless embedding may not be flat; however, every graph with a linkless embedding also has a flat embedding.

Does every flat embedding have a homeomorphic straight embedding? Two embeddings are *homeomorphic* if there is a continuous deformation of space that takes one embedding to the other. Not every linkless embedding has a homeomorphic straight embedding: for instance, an embedding of the triangle K_3 that ties it into a trefoil knot is linkless, but cannot be straightened (the simplest representation of the trefoil with straight edges requires six edges in the cycle). However, this example is not a flat embedding.

Analogously to Wagner's theorem for planar graphs, the linklessly embeddable graphs may be characterized by a set of seven forbidden graph minors (the *Petersen family*, which includes K_6 and the Petersen graph) [RST95]. Based on this characterization, it is possible to recognize linklessly embeddable graphs and find flat embeddings for them in linear time [KKM10]. For planar graphs, another important result that goes beyond recognition and embedding is Fáry's theorem, which states that if a graph has a noncrossing embedding in the plane with arbitrary curves (or polygonal chains) for its edges, then it also has a noncrossing embedding with straight line segments for its edges. This result underlies many graph drawing techniques, because straight-line edges are easier for computers to draw and easier for humans to read.

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Covering points with rectangles
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Given a set S of n points and an integer $k \leq n$, how efficiently can we find the axis-aligned rectangle of minimum area that covers $n - k$ points of S, that is, all but k of the points? The motivation for the problem comes from clustering, where the k points to ignore are outliers which we would like to identify.

Several known algorithms solve this problem, with running times $O(n + k^3)$ [AB+11], $O(n +$

[2] http://mathoverflow.net/questions/29660/.

$k^2(n-k)$) [SK98], and $O((n-k)^2 n \log n)$ [AI+91] (when the rectangle can have any orientation). Observe that, unless $k = o(n)$ or $k = n - c$ for some constant c, all these algorithms run in cubic time. The question is whether subcubic time is possible for the general problem.

Several specializations of the problem render it much simpler. For example, if the aspect ratio of the rectangle is prescribed, the problem can be solved in $O(n \log n)$ time [Cha99]. Let L, R, T, B be the set of k leftmost, rightmost, topmost, and bottommost points of S, respectively. If $(L \cup R) \cap (T \cup B) = \emptyset$, each dimension can be solved independently, leading to an $O(n + k^2)$-time algorithm.

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Counting points in circles

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Given n equal-radius circles whose centers form the points of a regular $\sqrt{n} \times \sqrt{n}$ grid, and given n points in the plane, how quickly can we count the number of points in each circle? This problem can be solved, in the more general case where the circle centers are not constrained to form a grid, in $O^*(n^{4/3})$ time via batched circular range queries. But does the grid structure help at all?

A similar question arises by dualizing the problem: given n equal-radius circles whose centers form a regular grid, and given n points, count the number of circles containing each point.

Domatic partition problems

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The domatic partitioning problem asks to partition a graph into a maximum number of vertex-disjoint dominating sets. It is known to be NP-hard. The new problem is the *independent* domatic partition problem, which seeks to partition a graph into a maximum number of disjoint independent dominating sets. For more details, see [MLM10].

References

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Separating and covering points in the plane

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1. Let B and R be sets of n blue and n red points in the plane in general position (i.e., no three points are collinear). What is the minimum number $f(n)$ such that one can always find a simple polygon with at most $f(n)$ sides that separates the blue and red points (i.e., the blue points are inside and the red points are outside of the polygon)? It is known that $n \le f(n) \le 3\lceil \frac{n}{2} \rceil$. (Note that the problem is interesting only under the general position assumption, for if all the points were on a line in the order red, blue, red, blue, ..., then we would need at least a $2n$-gon to separate them.)

2a. What is the smallest number $g(n)$ such that any n points in the plane can be covered by a simple (non-self-intersecting) polygonal line with at most $g(n)$ sides? Only trivial bounds are known: $n/2 \le g(n) \le n$.

2b. What is the smallest number $h(n)$ such that any n points in the plane can be covered by a polygonal line (possibly self-intersecting) with at most $h(n)$ sides? The known bounds are $n/2 \le h(n) \le n/2 + o(n)$, where the lower bound is obvious, while the upper bound is obtained by repeatedly using the Erdős-Szekeres theorem. Thus the gap in this version is quite small.

Update to 2b. At the GWOP 2011 workshop, E. Welzl proposed the following nice version of the problem. Call a set of n points in the plane *perfect* if it can be covered by a polygonal line (possibly self-intersecting) with at most $\lceil n/2 \rceil$ sides. For example, a set of points in convex position is perfect. The problem is to determine the maximum number $p(n)$ such that any set of n points in the plane has a perfect subset of size $p(n)$. By the Erdős-Szekeres theorem, $p(n) = \Omega(\log n)$. Can this bound be improved?

Orthogonal layering
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Decompose a graph G into edge subsets $E_1, E_2, \ldots, E_k$ such that each $G[E_i]$ is planar and maximum degree 4. What is the minimum orthogonal thickness $\hat{\Theta}(G)$ of G? The poser conjectures that $\hat{\Theta}(G) \leq \lceil \Delta/4 \rceil + 1$, where Δ is the maximum degree of G. See his paper [TH10] for more details.

References

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Largest independent set in rectangle-Delaunay
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Define a *rectangle-Delaunay graph* for a set of n points in the plane (no two on a horizontal or vertical line) by connecting any two points that are opposite corners of an empty axis-parallel rectangle. This graph can have a quadratic number of edges. What is the size of the largest independent set in this graph, as a worst-case function of n?

This problem is related to conflict-free colorings. Erdős-Szekeres yields a lower bound of $\Omega(\sqrt{n})$, which the poser improved to $\Omega(n^{0.618})$. For random points in a square, Chen et al. [CPZT09] established an upper bound of $O(n(\log\log n)^2/\log n)$. (And this bound is nearly tight for random points in a square.) The poser conjectures $n/\operatorname{polylog} n$ is the right bound.

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