# Single-Player and Two-Player Buttons & Scissors Games

Kyle Burke[1] Erik D. Demaine[2] Robert A. Hearn[3] Adam Hesterberg[4] Michael Hoffmann[5] Hiro Ito[6] Irina Kostitsyna[7] Maarten Löffler[8] Yushi Uno[9] Christiane Schmidt[10] Ryuhei Uehara[11] Aaron Williams[12]

## Abstract

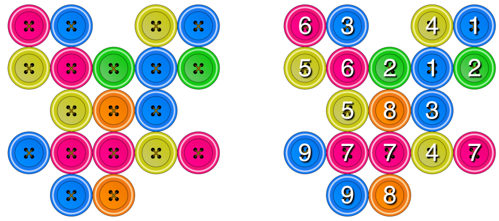
The Buttons & Scissors puzzle was recently shown to be NP-hard. In this paper we continue studying the complexity of various versions of the puzzle. For example, we show that it is NP-hard when the puzzle consists of $C = 2$ colors, and polytime solvable for $C = 1$. Similarly, it is NP-hard when each color is used by at most $F = 4$ buttons, and polytime solvable for $F = 3$. We also consider restrictions on the board size, cut directions, and cut lengths. Finally, we introduce new two-player games and show that they are PSPACE-complete.

## 1 Introduction

Buttons & Scissors is a single-player puzzle by KyWorks that was recently studied by Gregg et al. [2]. A *level* is an $n \times n$ grid of buttons of different colors; each position is either empty or has a button of a single color sewn to it. The goal is to remove all buttons using horizontal, vertical and diagonal scissor cuts. A cut is *feasible* if it removes at least two buttons of the same color and no buttons of any other color. Figures 1(a)–(b) show a sample level and solution. See [2] for further clarification of the rules and terminology.

Deciding whether a level is solvable is NP-complete [2]. We show that several restricted versions of the puzzle remain hard, and provide polytime algorithms for a number of easier versions. We also introduce two-player Buttons & Scissors games and show that they are PSPACE-complete. Due to space restrictions, most proofs are sketched or omitted.

[1]Plymouth State University, kgburke@plymouth.edu
[2]Massachusetts Institute of Technology, edemaine@mit.edu
[3]bob@hearn.to
[4]Massachusetts Institute of Technology, achester@mit.edu
[5]ETH Zürich, hoffmann@inf.ethz.ch
[6]The University of Electro-Communications, itohiro@uec.ac.jp
[7]Technische Universiteit Eindhoven, i.kostitsyna@tue.nl. Supported in part by NWO project no. 639.023.208.
[8]Universiteit Utrecht, m.loffler@uu.nl
[9]Osaka Prefecture University, uno@mi.s.osakafu-u.ac.jp
[10]The Hebrew University of Jerusalem, Israel, cschmidt@cs.huji.ac.il. Supported by the Israeli Centers of Research Excellence (I-CORE) program (Center No. 4/11).
[11]Japan Advanced Institute of Science and Technology, uehara@jaist.ac.jp
[12]Bard College at Simon's Rock, awilliams@simons-rock.edu




(a) (b)

Figure 1: (a) Level 6 in the Buttons & Scissors app is on a $5 \times 5$ grid with $C = 5$ colors, each used at most $F = 6$ times; (b) a solution using nine cuts of minimum length $\ell = 2$ and all four directions ($d = \ast$).

## 2 Notation

Each Buttons & Scissors level can be parameterized as follows (see Figure 1 for an example):

1. The *board size* $m \times n$.
2. The *number of colors* $C$.
3. The *maximum frequency* $F$ of an individual color.
4. The *cut directions* $d$ can be limited from the original four directions, which we denote by $\ast$. We only consider $d \in \{\ast, \times, +, \nearrow, -\}$ because an $n \times m$ board can be rotated 90° to an equivalent $m \times n$ board, or 45° to an equivalent $k \times k$ board for $k = n + m - 1$ by adding blank squares.
5. The *cut length* $\ell$ is the minimum number of buttons required to be removed by a feasible cut.

These parameters give the following decision problem (the original problem is $B\&S[n \times n, \infty, \infty, \ast, 2](B)$):

**Decision Problem:** $B\&S[m \times n, C, F, d, \ell](B)$.
**Input:** Given an $m \times n$ board $B$ with buttons of $C$ colors, where each color is used at most $F$ times.
**Output:** True if $B$ has a solution with minimum cut length $\ell$ using $d$ directions. Otherwise, False.

## 3 Single-Player Puzzle

We now present our results on the single-player puzzle.

### 3.1 Board Size

**Remark 1** *If a Buttons & Scissors board can be solved, then it can be solved using only cuts of* 2 *or* 3 *buttons.*

**Theorem 1** *Buttons & Scissors is polytime solvable on $1 \times m$ boards. That is, $B\&S[1 \times n, \infty, \infty, -, 2](B) \in P$.*

**Proof.** Consider the following context-free grammar,

$$S \to \epsilon \quad S \to S\Box \quad S \to xSx \quad S \to xSxx$$
$$S \to SS \quad S \to \Box S \quad S \to xxSx$$

where $\Box$ is an empty square and $x \in \{1, 2, \ldots, C\}$. By Remark 1, the solvable $1 \times m$ boards are in one-to-one correspondence with the strings in this language. $\Box$

**Theorem 2** *Given a full $2 \times m$ Buttons & Scissors board with $C = 2$ and a constant $s$, there exists a polynomial time algorithm that removes all but $s$ buttons from the board with feasible cuts.*

### 3.2 Number of Colors

**Theorem 3** *Buttons & Scissors is polytime solvable for 1-color, and is NP-complete for 2-colors. That is, $B\&S[n \times n, 1, \infty, *, 2](B) \in P$ and $B\&S[n \times n, 2, \infty, *, 2](B)$ is NP-complete.*

**Proof Sketch:** In a graph $G$, the maximum number of vertices that can be covered by edge-disjoint $K_2$ and $K_3$ subgraphs is polytime computable in the size of $G$ (see Cornuéjols et al. [1]). We convert each 1-color board $B$ into a graph whose vertices can be perfectly covered if and only if $B$ is solvable. The transformation uses Remark 1 and is non-trivial since the order of cuts is important. The 2-color reduction is in the full paper. $\Box$

### 3.3 Frequency of Colors

**Remark 2** *If board $B'$ is obtained from board $B$ by removing every button of a single color, then $B\&S[m \times n, C, F, d, \ell](B) \implies B\&S[m \times n, C, F, d, \ell](B')$ (i.e., it is impossible that Buttons & Scissors is solvable on $B$, but not solvable on $B'$ with the same parameters).*

**Theorem 4** *Buttons & Scissors is polytime solvable for maximum color frequency $F = 3$, and is NP-complete for $F = 4$. That is, $B\&S[n \times n, \infty, 3, *, 2](B) \in P$ and $B\&S[n \times n, \infty, 4, *, 2](B)$ is NP-complete.*

**Proof Sketch:** If $F = 3$, then each color is removed by a single cut in any solution. By Remark 2, these cuts cannot make a solvable board unsolvable. Thus, there is a simple greedy algorithm for deciding solvability.

It was proven that $B\&S[n \times n, \infty, 7, *, 2](B)$ is NP-complete via 3-SAT [2]. We use the same reduction for $F = 4$, with Figure 2 replacing each OR gadget. $\Box$

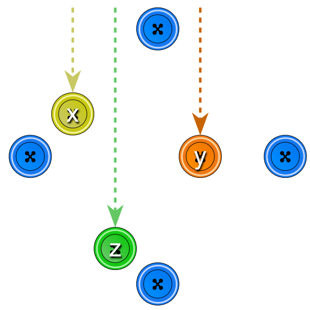



Figure 2: The left blue button can be removed if and only if "$x$ is removed $\vee$ $y$ is removed $\vee$ $z$ is removed".

### 3.4 Cut Directions

NP-completeness for horizontal and vertical cuts and $F = 7$ was proven in [2]. We improve this to $F = 6$.

**Theorem 5** *$B\&S[n \times n, \infty, 6, +, 2](B)$ is NP-complete.*

### 3.5 Cut Lengths

In Section 3.2 we saw that the 1-color version has a polytime algorithm. However, if the minimum cut length is set to $\ell = 3$ (instead of $\ell = 2$) then it is NP-complete.

**Theorem 6** *$B\&S[n \times n, 1, \infty, *, 3](B)$ is NP-complete.*

## 4 Two-Player Games

In our two-player games, the players take turns making feasible cuts on a common board $B$. In a *partisan game*, some cuts are available to one player, but not the other. In an *impartial game* all feasible cuts can be made by both players. We consider two losing conditions:

1. The player cannot execute a feasible cut (LAST).
2. The player removed fewer buttons (MAX).

**Theorem 7** *The partisan LAST two-player game is PSPACE-complete, where player 1 cuts blue buttons, player 2 cuts red, and both cut green (if any).*

**Theorem 8** *The impartial MAX and LAST two-player Buttons & Scissors games are PSPACE-complete.*

### References

[1] G. Cornuéjols, D. Hartvigsen, and W. Pullyblank. Packing subgraphs in a graph. *Operations Research Letters*, 1(4):139–143, 1982.

[2] H. Gregg, J. Leonard, A. Santiago, and A. Williams. Buttons & Scissors is NP-complete. In *Proc. 27th Canad. Conf. Comput. Geom.*, 2015.